E_∞-structures on homotopy colimits of commutative diagram space monoids

Master thesis

Thijs Heijligenberg

Supervised by Steffen Sagave

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Contents

1 Introduction

Algebraic topology is the study of spaces using the methods of algebra. Usually this means that one is interested in certain algebraic structures derived from a space, such as the (co)homology groups or homotopy groups, which one can study using the wide variety of methods from algebraic disciplines. In this text we will not study these algebraic objects themselves. The focus will, however, be on spaces having the same algebraic invariants. This means that we are interested in spaces which are either homotopy equivalent or weakly equivalent, although some maps we encounter may do better than this.

While algebraic topology is primarily the study of topological spaces, the theory often touches on different but related structures as well. For instance we will look at categories, and describe a functor which makes a space out of a category; this allows us to study a wide range of spaces by looking at their underlying categories. One of our primary objects of study will be diagrams of spaces (i.e. a sequence of spaces with certain maps between them), and how we can transform such a diagram into a space. We will also rely heavily on simplicial sets, as they are in a sense equivalent to topological spaces but much easier to analyze algebraically. We will also encounter bisimplicial sets, simplicial spaces, and topological categories to name a few, and study how they can be transformed into topological spaces.

We will pick up some tools to study these structures as well. For example we will define (co)ends as a convenient way to describe a lot of the constructions we will encounter, and we will give a very brief introduction to model categories as they give a lot of context to some definitions and results. These, together with the transformations between the structures mentioned above, will form the background to the results we are interested in.

The concept we want to study is that of a certain class of spaces which come equipped with a certain kind of binary operation. This operation is in our case associative and commutative up to homotopy; this notion of being almost commutative can be elegantly described using the so-called Barratt-Eccles operad, and we call these objects E_∞-spaces. We will describe a type of category, called a permutative category, which gives rise to E_∞-spaces. These form the contents of chapter 2.

We will then give the notion of an $\mathcal{I}$-space (which is nothing more than a functor from the category $\mathcal{I}$ to the category of spaces), and that of a homotopy colimit. The latter is, like the ordinary colimit, a construction which makes a functor into a space. The catch here is that the homotopy colimit of a commutative $\mathcal{I}$-space again has the structure of an E_∞-space. This is described in chapter 3.

Following that we give a construction which makes a commutative $\mathcal{I}$-space out of a permutative category, and we combine it with the results from chapter 3 to again get an E_∞-space. We then explore how these two E_∞-spaces we have gotten out of a permutative category are related. This is the content of chapter 4.

After that we give slight generalisation where we do not look at a permutative category $\mathcal{K}$ but rather at a functor from $\mathcal{K}$ to the category of spaces. This is the final chapter 5.

The main result of the text up to chapter 4 is that the E_∞-structure on the classifying space of a permutative category is equivalent to the E_∞-structure on the homotopy colimit of a suitable $\mathcal{I}$-space. This has been described in the literature before in chapter 7 of [SS16]. The main result of Chapter 5 is a generalisation of this; for a commutative category $\mathcal{K}$ the E_∞-structure on the homotopy colimit of a $\mathcal{K}$-diagram of spaces is equivalent to the E_∞-structure on a suitable $\mathcal{I}$-space. The importance of this fact is that it shows how central to the theory the category $\mathcal{I}$ is; any homotopy colimit with an E_∞-structure is equivalent to a homotopy colimit of an $\mathcal{I}$-space.

1.1 Conventions

All categories are locally small, and arbitrary categories will tacitly be assumed to be (co)complete as well. This means that in statements like "Let $\mathcal{C}$ be a category. Define ..." we assume $\mathcal{C}$ to be locally small and (co)complete. Specified categories such as the category $\mathcal{I}$ we study in chapter 3 may not be (co)complete however. The category of small categories is denoted **cat**, and the category of topological spaces is denoted **Top**.

Morphism sets in a category $\mathcal{C}$ will be denoted $\mathcal{C}(-,-)$. The category of functors from $\mathcal{C}$ to $\mathcal{D}$ will be denoted $\mathcal{D}^{\mathcal{C}}$, and natural transformation sets will be denoted $\mathcal{D}^{\mathcal{C}}(-,-)$.

Isomorphisms are denoted $\cong$, and weak equivalences are denoted $\simeq$ or mentioned explicitly.

The set $\mathbb{N}$ contains the number 0.

2 The Barratt-Eccles operad

In this section we will start with some necessary underlying concepts which can be found in numerous books and other sources. The primary source that was used here is [Ric20].

2.1 Monoidal categories

In what follows, almost all categories we will be interested in will have some product-like structure, which we will call *monoidal* structures. The definition of a monoid in the classical sense can then be generalised to this setting. Monoidal structures can be formed by products or coproducts alike, and even composition of endofunctors gives a monoidal structure. Note that in the definition the symbol $\otimes$ is used, as is common in a lot of the literature; this should not be confused with the tensor product of modules or in other categories. These operations will form a monoidal structure, but any additional properties they possess need not hold in other categories.

Definition 2.1. A **monoidal** category $\mathcal{C}$ is a category with a bifunctor $-\otimes- : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, an object $e\in\mathcal{C}$, and natural isomorphisms for all objects C_1, C_2, C_3 :

$$C_1\otimes(C_2\otimes C_3)\cong(C_1\otimes C_2)\otimes C_3, C\otimes e\cong e\otimes C\cong C$$

These isomorphisms have to satisfy certain coherence condition which we will not give here. A full definition can be found in Definition VIII.1.4 of [Ric20].

If the isomorphisms above are equalities we have a **strict monoidal** category. A monoidal category is **symmetric** if it additionally comes equipped with natural isomorphisms

$$\tau : C\otimes D\to D\otimes C$$

satisfying $\tau^2 = 1$, $\tau_{C,e} = 1$ and properties concerning associativity. If the symmetry maps are equalities, we call a category **strictly symmetric monoidal**. A category which is strictly monoidal but non-strictly symmetric is called **permutative**.

Examples:

- The category of sets has a monoidal structure by both the product and disjoint union. The unit is in the first case a one-element set, and in the second the empty set. Here we see that the first example is not strict. In general, categories with finite products and a terminal object form a monoidal category. Dually, the same holds for finite coproducts and an initial object. This can of course be extended to the category of presheaves by doing this pointwise.

- The category of small categories has a monoidal structure by the product category and the one-point category. This is a special case of the above, but will be of importance later.

- In R-modules for a commutative ring R, the tensor product gives a monoidal structure with unit R. This specialises to vector spaces and abelian groups
- The category of endofunctors of a category has a strict monoidal structure by composition of functors. The identity functor forms a unit.
- The category of simplicial sets which will be introduced in the next section has the pointwise product as simplicial structure: $(A \otimes B)_n = A_n \times B_n$. The unit is the simplicial set with one element in each degree.

Of course, this would not be category theory if there were not some concept for mappings respecting this kind of structure:

Definition 2.2. Let $(\mathcal{C}, \otimes, e)$ and $(\mathcal{D}, \oplus, d)$ be monoidal categories. A **lax monoidal** functor $F : \mathcal{C} \to \mathcal{D}$ is a functor with natural maps

$$\phi : F(C) \oplus F(C') \to F(C \otimes C')$$

and a map $e' \to F(e)$, which are compatible with the associative and unital structures. The functor is called **strong monoidal** if ϕ is a natural isomorphism, and **strictly monoidal** if it is an equality. If $\mathcal{C}$ and $\mathcal{D}$ are symmetric with symmetry isomorphisms τ and τ' respecitively we say that F is **lax symmetric monoidal** if F is lax monoidal and the following diagram commutes:

$$\begin{tikzcd} F(A) \oplus F(B) \arrow[r, "\tau'"] \arrow[d, "\phi"] & F(B) \oplus F(A) \arrow[d, "\phi"] \\ F(A \otimes B) \arrow[r, "F(\tau)"] & F(B \otimes A) \end{tikzcd}$$

If this diagram commutes and F is strong or strictly monoidal then we say that F is **strong** or **strict symmetric monoidal**.

We can now define the concept of a monoid, which will play a central part in the theory at hand.

Definition 2.3. A **monoid** in a monoidal category $(\mathcal{C}, \otimes, e)$ is an object M with a morphism $\mu : M \otimes M \to M$ and $e \to M$ satisfying associativity and unitality. In a symmetric monoidal category with symmetry map τ we further impose $\mu \circ \tau = \mu$ for a **commutative monoid**.

Examples:

- A monoid in Sets with $\times$ is what is classically a monoid, and commutative monoids are what is usually understood as a commutative monoid.
- A monoid in the category of k-modules is a k-algebra, where k is a field.
- A monoid in the category of endofunctors of a category is called a **monad**. Explicitly, it is a functor $M : \mathcal{C} \to \mathcal{C}$ with a natural transformation $M \circ M \Rightarrow M$, i.e. for every X a morphism $M(M(X)) \to M(X)$ which is natural in X, and also a map $X \to M(X)$. An example of a monad is the powerset operation on sets, with the "multiplication" given by the union.

Monads will turn out to be of use to us, as the operads we will form later can be realised as monads. Also of use to us is the concept of an algebra over a monad:

Definition 2.4. Let M be a monad on a category $\mathcal{C}$ with multiplication μ and unit η. An **algebra** over M is an object X of $\mathcal{C}$ with a morphism $f : M(X) \to X$ satisfying $f \circ M(f) = f \circ \mu$ and $f \circ \eta = \text{id}$. A **morphism of algebras** between (X, f) and (Y, g) is a map $F : X \to Y$ such that $g \circ M(F) = F \circ f$.

The category of algebras over a monad M in a category $\mathcal{C}$ is denoted $\mathcal{C}[M]$.

The definition of a monad can also in general be given for other monoids, where such a structure is generally called a **module**. To do this, view the object X from the algebra as a constant endofunctor. Then the morphism f given in the definition of the algebra can be viewed as a map $M \otimes X \to X$ where $\otimes$ is the monoidal structure on endofunctors. For general monoids A we can then say that an object Y is a module if there is a morphism $A \otimes Y \to Y$ satisfying some conditions corresponding to those imposed on an algebra. We will not use this more general concept in this text.

2.2 Classifying spaces

Let $\mathcal{C}$ be a small category. We want to associate to this a topological space, which will be denoted $B\mathcal{C}$. Since we know a lot of constructions for topological spaces we can immediately make definitions for small categories using this topological space, e.g. $\pi_n(\mathcal{C}, x) = \pi_n(B\mathcal{C}, x')$ for some particular $x' \in B\mathcal{C}$. We can also study how certain concepts such as (co)limits transfer from categories to topological spaces. Finally it allows us to define weak equivalences and homotopy equivalences for categories.

Definition 2.5. Let Δ be the category of finite sets $[n] = \{0, ..., n\}$ for $n \geq 0$ and order-preserving functions. A *simplicial object* in a category $\mathcal{C}$ is a a contravariant functor from Δ to $\mathcal{C}$. For A a simplicial object in a category we will often abbreviate $A([n])$ by A_n. The simplicial objects in a category form a category of their own where the morphisms are the natural transformations, and we denote this category $s\mathcal{C}$.

Defining a simplicial object thus means giving a morphism for each of the morphisms in Δ in a functorial manner. In practice it suffices to only look at a set of generating morphisms in Δ:

Lemma 2.6. *Let $d_i : [n-1] \to [n]$ and $s_i : [n] \to [n-1]$ be the maps which respectively are injective and miss i, and are surjective and hit i twice. Explicitly, we have for $d_i(j) = j$ if $j < i$, and $d_i(j) = j+1$ if $j \geq i$, and a similar definition for s_i.*

All maps in Δ are generated by these maps. Consequently, for a simplicial set A it suffices to give the sets A_n and the maps $A(d_i)$ and $A(s_i)$ satisfying certain relations.

Remark 2.7. We will mainly be concerned with the category of simplicial sets from this point on. This category is in some sense equivalent to the category of topological spaces, and its main advantage is that it will be more convenient to talk about algebraic structures in this setting as it has a more combinatorial nature. Later on we will also encounter bisimplicial sets, which are simplicial objects in the category of simplicial sets. Using the adjunction between products and functor categories we can transform these to

$$\mathbf{cat}(\Delta^{op}, \mathbf{cat}(\Delta^{op}, \mathrm{Set})) \cong \mathbf{cat}(\Delta^{op} \times \Delta^{op}, \mathrm{Set}) = \mathbf{cat}((\Delta \times \Delta)^{op}, \mathrm{Set})$$

It is therefore safe to just write these as a functor with two indices.

Let us define the categories $[n]$ with objects $\{0, 1, 2...n\}$ and a unique morphisms $i \to j$ if $i < j$. We can define the the so-called face maps

$$\delta_j : [n] \to [n+1], \delta_j(i) = \begin{cases} i & \text{if } i \leq j \\ i+1 & \text{else} \end{cases}$$

and degeneracy maps

$$\sigma_j : [n] \to [n+1], \sigma_j(i) = \begin{cases} i & \text{if } i \leq j \\ i-1 & \text{else} \end{cases}$$

To make these functors we also need to give an action on morphisms. However, there is exactly one morphism between any two objects so this is uniquely determined. More explicitly, δ_j inserts an identity and σ_j composes two maps.

Definition 2.8. The **nerve** of $\mathcal{C}$, denoted $N\mathcal{C}$, is given by $N\mathcal{C}_n = \mathbf{cat}([n], \mathcal{C})$. The face and degeneracy maps are defined by $d_i(f) = f \circ \delta_i$ and $s_i(f) = f \circ \sigma_i$.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor we get a morphism of simplicial sets $NF : N\mathcal{C} \to N\mathcal{D}$ defined by $f \mapsto f \circ F$. More explicitly, if we think of an element of the nerve not as a functor but as its image we can think of an element of $N\mathcal{C}_n$ as a chain $c_0 \xrightarrow{f_1} ... \to c_n$ of length n. The image of this under NF would then be $F(c_0) \xrightarrow{F(f_1)} ... \to F(c_n)$

These face and degeneracy maps have a very explicit construction, by the above remark on what the δ and σ do on maps, and the fact that objects of the nerve must be functors. If we have a chain

$$c_0 \xrightarrow{f_1} c_1 ... \xrightarrow{f_n} c_n$$

and we apply s_i and d_i to it we get the two chains

$$c_0 \to ... c_i \xrightarrow{\mathrm{id}} c_i \to ... \to c_n, c_0 \to ... \to c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \to ... \to c_n$$

Note that it is really necessary for a category $\mathcal{C}$ to be small to take the nerve, as $N\mathcal{C}_0 = \mathrm{Obj}(\mathcal{C})$ needs to be a set. We will see some examples of this construction after we have defined how to translate simplicial sets to topological spaces.

Lemma 2.9. *The category of simplicial sets is complete and cocomplete.*

Proof. This can generally be done for functor categories, as we can take these limits pointwise. The statement then follows because the category of sets is complete and cocomplete. □

This means we can give the category of simplicial sets a monoidal structure using both the product and coproduct. The units in this case are the one-point simplicial set $*$ with $*([n]) = \{*\}$, and the empty simplicial set.

Proposition 2.10. *The nerve functor is strong monoidal with respect to the product.*

Proof. We can, given $\mathcal{C}, \mathcal{D}$ small categories, transform

$$N(\mathcal{C} \times \mathcal{D})_n$$
$$= \{C_1 \times D_1 \to ... \to C_n \times D_n | C_i \in \mathcal{C}, D_i \in \mathcal{D}\}$$

$$cong\{C_1 \to ... \to C_n | C_i \in \mathcal{C}\} \times \{D_1 \to ... \to D_n | D_i \in \mathcal{D}\}$$
$$= (N\mathcal{C} \otimes N\mathcal{D})_n$$

Maps in $\mathcal{C} \times \mathcal{D}$ are pairs of maps in $\mathcal{C}$ and $\mathcal{D}$ by definition, so we can project these map pairs to their first or second coordinate to get the resulting maps in $N\mathcal{C}$ and $N\mathcal{D}$. To go the other way one can take the product of maps and objects, and these operations are clearly inverse to each other. □

Proposition 2.11. *The nerve functor is strong monoidal with respect to the coproduct.*

Proof. A chain in a coproduct of categories $\mathcal{C} \sqcup \mathcal{D}$ has a first entry x in either $\mathcal{C}$ or $\mathcal{D}$, and all chains starting in x stay within this component, and are thus either chains within $\mathcal{C}$ or $\mathcal{D}$. □

The next point in our exposition is giving a way to make a topological space out of a simplicial set. Simplicial sets have a very convenient algebraic definition, but topological spaces have some advantages as well. They are for instance easier to visualize in some cases, and definitions such as contractibility are more familiar in this setting.

Definition 2.12. Let S be a simplicial set with face maps σ_n and degeneracy maps δ_n. Consider the sets

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | \forall i t_i \geq 0, t_0 + ... + t_n = 1\}$$

called the standard n-simplices. They will carry the subspace topology and come equipped with standard face and degeneracy maps d, s. The **geometric realization** of S is the topological space $\coprod S_n \times \Delta^n / \sim$ where S_n carries the discrete topology. The equivalence relation is generated by

$$(\delta_i(x), t) \sim (x, d_i(t)), (\sigma_i(x), t) \sim (x, s_i(t))$$

This also forms a functor, so for a morphism of simplicial sets $f : X \to Y$ we get $|f| : |X| \to |Y|$, which is defined by $|f|(x,(t_0,...,t_n)) = (f(x),(t_0,...,t_n))$. The fact that this is well-defined with respect to the equivalence relation comes from the fact that f is a morphism of simplicial sets.

Proposition 2.13. *The geometric realization functor is lax monoidal with respect to the product of simplicial sets and the compactly generated product of topological spaces.*

See [Ric20] for a constructive proof for $|X \times Y| \cong |X| \times |Y|$ whenever $|X| \times |Y|$ is a CW-complex. In general this statement is false when talking about the ordinary product of topological spaces; the remedy to this is to take the compactly generated product of topological spaces. A proof for this case can be found in Remark X.6.9 of [Ric20], and the subtleties of compactly generated products are explained for example in section VIII.5 of [Ric20]. The full details of both proofs are noticeably more involved than the proofs for the nerve functor, and this background is not necessary for the scope of this text.

Definition 2.14. The **classifying space** of $\mathcal{C}$, denoted $B\mathcal{C}$, is $|N(\mathcal{C})|$. Composing these two functors gives that if we have a functor $F : \mathcal{C} \to \mathcal{D}$ we get $BF : B\mathcal{C} \to B\mathcal{D}$

The definition of a simplicial set as a 'sequence' of sets building on top of each other is reminiscent of the definition of a CW-complex, which is a sequence of skeleta which build on top of each other by attaching cells. This intuition is correct, and formalised in the statement below:

Proposition 2.15. *The classifying space of a small category has a filtration induced by the original category which exhibits the space as a CW-complex.*

Note that the proposition is strict in that it does not assume this to hold up to weak equivalence.

An immediate obstacle one finds when one want to prove this is that elements of the form $s_i(x)$, the so-called **degenerate** elements, act as copies of the element x. The proof becomes rather cumbersome and involves identifying every element with a non-degenerate element; an explicit proof can be found in [Ric20], X.6. We will see this idea in practice when we describe the realization of the Δ^n on the next page.

In what follows we will, as mentioned before, mostly work with simplicial sets. It is not true that the categories of simplicial sets and topological spaces are equivalent, although it is true that they are Quillen equivalent. Quillen equivalence is weaker than normal equivalence of categories but also more specific in a sense. We will later define what this means but we will not go into depth nor will we prove it. Something we can state already is the following:

Proposition 2.16. *Let S be the singular simplicial complex functor. Then* $|-| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : S$ *are adjoint functors.*

Proof. Deferred until a later point at section 3.2 $\square$

The statement that this is a Quillen equivalence says that these adjoint maps preserve quite some more structure, for example that the composition is naturally weakly equivalent to the identity on some objects. In the next section we will give an explicit definition for this.

Since we now have a method of conversion between simplicial sets and topological spaces we can state definitions we know from the classical treatment of topological spaces, and directly translate them to simplicial sets.

Definition 2.17. Let X be a simplicial set. We say X is **contractible** if $|X|$ is contractible.

In what follows we will use the simplicial set $\Delta^n : [m] \mapsto \Delta([m], [n])$. The degeneracy and face maps are given by compositions with the standard δ, σ. These form the analogue of the standard n-simplices of topological spaces. Indeed $|\Delta^n|$ is the standard n-simplex; let us look for example at Δ^2. In degree 0 it has three elements, as [0] has one element and [2] has three. There are 6 maps from [1] to [2], but three of those are degenerate (these are the constant maps), so we only get 3 vertices; when one writes out what the face maps give one finds out this forms the edge of a triangle. There is only one non-degenerate map from [2] to [2] (the identity) filling in the face of the triangle. All higher maps are degenerate. A triangle is a presentation of the standard 2-simplex.

Definition 2.18. Let $f, g : X \to Y$ be morphisms of simplicial sets. A **simplicial homotopy** between f and g is a map $H : X \times \Delta^1 \to Y$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}\times d_0"] \arrow[dr, "f"'] & X \times \Delta^1 \arrow[d, "H"] & X \arrow[l, "\mathrm{id}\times d_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

Here X is identified with $X \times \Delta_0$

This definition is very reminiscent of the definition of a homotopy in topological spaces. There one requires $H(x, 0) = f(x), H(x, 1) = g(x)$, but if one lifts the map $\mathrm{id} \times d_0$ to the level of topological spaces it is the map $(x, (\mathrm{id}, t)) \mapsto (x, (0, *)$ where id is the unique element of $(\Delta^1)_1$, and on the right $*$ is the unique element of the 0-simplex which is attached to the 0-map in $(\Delta^1)_0$. This means that $|H| \circ |(\mathrm{id} \times d_0)|(x, (\mathrm{id}, t)) = |H|(x, (0, *)) = |f|(x)$ which is the same as in the topological definition up to identifications. Here we used that id is the only non-degenerate element of $(\Delta^1)_1$.

The above identification gives an insightful relationship with the topological case, but is rather forgetful of much of the simplicial set Δ^1. A simplicial map should also take care of the degenerate elements, but luckily there is a very combinatorial interpretation of Δ^1. Since elements of $(\Delta^1)_n$ are order-preserving maps from $[n]$ to $[1] = \{0, 1\}$ it suffices to describe such an element x by the index at which $x(i-1) = 0, x(i) = 1$ since before that it is 0 and afterwards it will be 1. If we identify x with this i (where we put 0 if it is always 1 and $n+1$ if it never is) we can write $(\Delta^1)_n = [n+1]$. This means that giving a simplicial homotopy can be done by giving at level n a way of going from f_n to g_n in n steps.

Example 2.19. Let $\mathcal{C} = \{0 \xrightarrow{a} 1\}$ with a an isomorphism with inverse b. We prove that $N\mathcal{C}$ is contractible by giving a homotopy between a constant map to a point and the identity. The map to a point is given by $f_n(*) = 0 \to ... \to 0$. A homotopy H is given at level n by a map $N(\mathcal{C})_n \times [n+1] \to N(\mathcal{C})_n$ with

$$H_n(x, n+1) = f_n(*), H_n(x, 0) = x$$

We define $H_n(x, i)$ on $x = j_0 \xrightarrow{x_1} ... \xrightarrow{x_n} j_n$ by $0 \xrightarrow{\mathrm{id}} 0 \to ...0 \to j_i \xrightarrow{x_{i+1}} ...j_n$, so we collapse the maps one by one. We clearly have $H_n(x, 0) = x$ as nothing has changed there, and $H_n(x, n+1) = f_n(*)$ as all arrows have been changed to identities. One can write out and check that this gives a proper map of simplicial sets. This category and its classifying space will come up again at a later point, where we will identify $B\mathcal{C}$ with S^∞; we have thus proven that S^∞ is contractible.

Lemma 2.20. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between small categories. A natural transformation $\tau : F \Rightarrow G$ induces a homotopy from BF to BG.*

Proof. Note that the category $[1] = \{0 < 1\}$ with two objects and one morphisms has as classifying space the unit interval. It thus suffices to give a functor $h : \mathcal{C} \times [1] \to \mathcal{D}$ with $h(-, 0) = F, h(-, 1) = G$. We define h to do this on objects, and on morphisms we take, with $f : C \to C'$, $h(f, \mathrm{id}_0) = F(f), h(f, id_1) = G(f), h(f, 0 < 1) = \tau_{C'} \circ F(f) = G(f) \circ \tau_C$. This defines a functor and we now define a homotopy

$$B\mathcal{C} \times [0, 1] \cong B(\mathcal{C} \times [1]) \xrightarrow{B(h)} B\mathcal{D}$$

□

Corollary 2.21. *A pair of adjoint functors F, G between $\mathcal{C}$ and $\mathcal{D}$ gives a homotopy equivalence between $B\mathcal{C}$ and $B\mathcal{D}$.*

Proof. We have the unit $\eta : \mathrm{id} \Rightarrow GF$ and counit $\epsilon : FG \Rightarrow \mathrm{id}$ inducing homotopies $\mathrm{id} \Rightarrow B(GF) = B(G)B(F)$ and vice versa. This gives that $B(F)$ and $B(G)$ form a homotopy equivalence. □

Corollary 2.22. *Equivalence of categories implies homotopy equivalence of their nerves/classifying spaces.*

An equivalence of categories is a somewhat stronger condition than we strictly need. An equivalence of categories is given by a pair of maps whose compositions are naturally isomorphic to the identity; to form the required homotopies we do not need those natural transformations to be natural isomorphisms. We see this for example in the following:

Proposition 2.23. *Let $\mathcal{C}$ be a small category with an initial object. Then $B\mathcal{C}$ is contractible.*

Proof. Let 0 be the initial object. We have $\{*\} \xrightarrow{F} \mathcal{C} \xrightarrow{G} \{*\}$, $F(*) = 0, G(x) = *$. The composition $G \circ F$ is the identity (as it is the only endofunctor of the one-point category), and $F \circ G$ is the constant 0 map. This is a natural transformation to the identity by the transformation formed by the unique maps out of 0. □

The contractibility of the space at Example 2.19 follows more easily now; the category $\mathcal{C}$ there is equivalent to the one-point (terminal) category for all n as any endofunctor on it is pointwise isomorphic to the identity as all objects are isomorphic.

2.3 Model categories

In this section we will give a very brief introduction to the definition of a model category. It will not be used extensively, but it will at times be useful to know what some of the terms mean for topological spaces or simplicial sets. Additionally this makes it easier to put some of the constructions into a broader context.

Definition 2.24. Let $\mathcal{C}$ be a complete and cocomplete category. A **model structure** on $\mathcal{C}$ consists of three classes of morphisms: the **fibrations**, **cofibrations**, and **weak equivalences**, such that the following properties hold:

1. If $f = g \circ h$, then if two out of three are in one of the distinguished classes the so is the third.

2. all three classes are closed under retracts: if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ C \arrow[r] & D \arrow[r] & C \end{tikzcd}$$

   is commutative with the horizontal rows composing to the identity, then if g is in one of the classes then so is f.

3. If

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

is a commutative square with f a cofibration, g a fibration and either f or g a weak equivalence, the the dotted arrow exists making both resulting diagrams commute.

4. Any map f may be factored as $f = g \circ h$ with f a cofibration, g a fibration and one of the two a weak equivalence.

We call morphisms that are fibrations and weak equivalences the **trivial fibrations** and those that are cofibrations and weak equivalences the **trivial cofibrations**. An object $c \in \mathrm{Obj}(\mathcal{C})$ is called **fibrant** if the map $X \to \mathbf{1}$, with $\mathbf{1}$ the terminal object, is a fibration. Likewise it is called **cofibrant** if the map $\mathbf{0} \to X$ is a cofibration.

Property 3 of the definition can be formulated as "cofibrations have left lifts with respect to trivial fibrations" and "fibrations have right lifts with respect to trivial cofibrations". The converse holds as well (i.e. morphisms which have left lifts with respect to trivial fibrations are cofibrations), which means it suffices to give either the class of fibrations or cofibrations to define both. Nevertheless it remains useful to give an explicit description of both classes in practical cases. We will not give precise description nor a proof of this fact; for a more thorough introduction the reader is referred to [DS95].

Example 2.25. In the classical model structure on the category of topological spaces, the weak equivalences are given by the weak homotopy equivalences. Fibrations are Serre fibrations; these are maps which have the left lifting property with respect to all the inclusions $D^n \to D^n \times I$, where D^n is the n-disk and I is the interval. Cofibrations are retracts of maps which arise by cell attachments.

From this it follows that cofibrant objects are spaces which are retracts of cell complexes; for $\emptyset \to X$ to be a retract of $A \to B$ we must have that $A = \emptyset$. All objects are fibrant, as the map $X \to *$ has all required lifts; given a map $D^n \to X$ we can make a map $D^n \times I \to X$ which is constant with respect to the interval.

Example 2.26. The classical model structure on simplicial sets has as weak equivalences the maps f for which the realization $|f|$ is a weak equivalence. The cofibrations are the objectwise injective morphisms. For fibrations we define the **horn** to be the simplicial sets $\Lambda^(k, n)$ to be simplicial subset of Δ^n gotten by taking all the order-preserving maps from Δ^n which do not have k in their image. This can be seen as taking the union of all but one of the faces. A fibration is then a map which has left lifts with respect to all the inclusions $\Lambda^(k, n) \to \Delta^n$, which is reminiscent of the definition of a Serre fibration as seen above.

In this case all objects are cofibrant, as the map $\emptyset \to X_n$ is injective for all n. The fibrant objects are what are known as Kan complexes, and we will not study these here.

Definition 2.27. Let $\mathcal{C}$ be a model category and $X \in \mathrm{Obj}(\mathcal{C})$. We define QX as a factorisation of the map $\mathbf{0} \to X$ as $\mathbf{0} \to QX \xrightarrow{p} X$ with p a trivial cofibration. We define RX as occurring in $X \to RX \to \mathbf{1}$ by factoring the map to the terminal object. If X is cofibrant we put $QX = X$, and if X is fibrant we put $RX = X$. Such maps are called **(co)fibrant replacements**

Example 2.28. If $\mathcal{S}$ denotes the model category of either topological spaces or simplicial sets, then in the category of $\mathcal{C}$-spaces we will be able to define a cofibrant replacement by $X \mapsto B(\mathcal{C}, \mathcal{C}, X)$ which is the bar construction we will see in Definition 4.14 and study in the subsequent section.

Definition 2.29. Let $X \in \mathrm{Obj}(\mathcal{C})$ an object of a model category, and $[\mathrm{id}, \mathrm{id}] : X \amalg X \to X$ the map induced by the identities. We call $C(X)$ a **cylinder object** of X if it fits in a factorisation $X \amalg X \to C(X) \xrightarrow{\sim} X$. Maps $f, g : X \to Y$ are **left homotopic** if there is a map $H : C(X) \to Y$ such that, if we denote $[i_0, i_1] : X \amalg X \to C(X)$ for the structure map, we have $H \circ [i_0, i_1] = [f, g]$.

Similarly, a **path object** is an object $P(X)$ with a factorisation of $\mathrm{id} \times \mathrm{id} : X \xrightarrow{\sim} P(X) \to X \times X$, and maps are **right homotopic** if there is a map $H' : X \to P(X)$ such that $H' \circ (j_0 \times j_1) = f \times g$.

If X is cofibrant, left homotopy defines an equivalence relation on $\mathcal{C}(X, Y)$, and if X is fibrant we get an equivalence relation on $\mathcal{C}(Y, X)$; furthermore if X is cofibrant and Y fibrant the equivalence relations on $\mathcal{C}(X, Y)$ agree, and the set of equivalence classes is denoted $\pi(X, Y)$. Under certain conditions we also get that composition is a well-defined map on homotopy classes. The details of this can be read in [DS95]. We can thus define the following:

Definition 2.30. Let $\mathcal{C}$ be a model category. We define the **homotopy category** of $\mathcal{C}$, denoted $\mathrm{Ho}(\mathcal{C})$, by

$$\mathrm{Obj}(\mathrm{Ho}(\mathcal{C})) = \mathrm{Obj}(\mathcal{C}), \mathrm{Ho}(\mathcal{C})(X, Y) = \pi(RQX, RQY)$$

There is a map $q : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, which is just the identity on objects and which takes homotopy classes on maps. One can prove this is a functor.

Definition 2.31. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with $\mathcal{C}$ a model category with $q : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ the aforementioned functor. Then the **left derived functor** of F is a functor LF with a natural weak equivalence $LF \circ q \Rightarrow F$ which is terminal in this situation. This means that for any other $L'F \circ q \Rightarrow F$ we have a factorisation through $LF \circ q$. Putting this graphically means that we want a (not necessarily commutative) diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"'] & \mathcal{D} \\ \mathrm{Ho}(\mathcal{C}) \arrow[ur, "LF"] & \end{tikzcd}$$

which is a best approximation to make this commute.

The **right derived functor**, denoted RF, is defined analogously except we now get an initial $F \Rightarrow RF \circ q$.

We will encounter this notion of a closest approximation to a commutative diagram again in the form of **Kan extensions** at Definition 3.2, which we will study in more detail. The situation there is quite different to the one we have here, as there we will define Kan extensions using (co)limits, which we do not have at our disposal in most homotopy categories.

The prime example we will see later is the homotopy colimit, which will be the left derived functor of the ordinary colimit; the main purpose of this section is to provide a background for this fact. We will however not prove this fact. The main obstacle for a deeper treatment of this material is that the category of diagrams $\mathcal{S}^{\mathcal{C}}$ for some indexing category $\mathcal{C}$ does have several choices for a model structure, but that the theory of these structures is quite deep, and not useful for the scope of this text. For a review of these model structures and their relation to homotopy colimits the reader is referred to [SS12] and [Gam10].

A convenient method to define left derived functors is given by the following:

Lemma 2.32. *Suppose F sends weak equivalences between cofibrant objects to isomorphisms. Let Q be the cofibrant replacement functor. Then $LF(X) = F(Q(X))$ defines a left derived functor of F.*

Proof. See [GJ09], Remark 8.4. □

Left derived functors between model categories are often also composed with the map to the homotopy category of the image. To be specific, when is a functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{C}, \mathcal{D}$ model categories we often mean $LF : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ where we implicitly derive the composite $\mathcal{C} \to \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$.

Proposition 2.33. *Let F be left adjoint to G and suppose F preserves weak equivalences between cofibrant objects and G preserves weak equivalences between fibrant objects. Then LF, RG exist and they are adjoint.*

Secondly, suppose that F and G are as above, and additionally that for X cofibrant, Y fibrant we have that $X \to GY$ is a weak equivalence iff its adjoint $FX \to Y$ is. Then the adjoint functors above form an equivalence of categories between $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$.

For a proof see 8.7 and 8.8 of [GJ09]. A pair of functors satisfying the conditions of the first part are called a **Quillen adjunction**. If the pair satisfies the second part it is called a **Quillen equivalence**. Quillen equivalences are the right kind of equivalences between model categories in a sense, in that it allows us to make comparisons such as the following:

Example 2.34. The model category of (compactly generated and weak Hausdorff) topological spaces is Quillen equivalent to the model category of simplicial sets. The functors are given by the topological realization $| - |$ and $\mathrm{Sing}(-) = \mathbf{Top}(\Delta^{\cdot}, -)$. This signifies that Quillen equivalence is weaker than normal equivalence of categories; we can prove that every topological space is weakly equivalent to the realization of a simplicial set (which forms a CW complex), but there is no way to make this a homeomorphism.

2.4 Operads

Let $(\mathcal{C}, \otimes, e)$ be a symmetric monoidal category. In this section we will introduce operads, which are a concept allowing us to quickly encapsulate all the relevant data to encode multiplicative structure in some situations. Roughly speaking an operad encodes the level of coherence of an operation. Coherence can mean associativity, which is the lowest level of coherence we will allow, or commutativity, which is the strictest we can get. Often however we have something which is in between. In our context we look at objects in a topological setting so it is naturally interesting to look at operations which are commutative up to homotopy or some other notion of weakness. It would of course be more convenient to have actual commutativity, but some important operations turn out not to be commutative.

The operads we define are sometimes called Σ-operads or symmetric operads. In those contexts an operad which is non-Σ or non-symmetric is one which does not have the so-called equidistributivity property which we will define below. In a later section we will give a small taste of what these non-symmetric operads could encode, but since it is not something we are interested in we will assume our operads to be symmetric.

Definition 2.35. An operad O is a collection of objects in $\mathcal{C}$ indexed by $\mathbb{N}$, where the symmetric group Σ_n actso on $O(n)$. It has a morphism $\eta : e \to O(1)$, and for $n = k_1 + ... + k_r$ a morphism $O(r) \otimes O(k_1) \otimes ... \otimes O(k_r) \to O(n)$, which we will usually denote by γ. These maps must satisfy some properties.

1. The first property is associativity: in the category of sets or spaces this can be written

   $$\gamma(x, \gamma(y_1, z_1^1, ..., z_{n_1}^1), .., \gamma(y_m, z_1^m, ..., z_{n_m}^m) = \gamma(\gamma(x, y_1, ..., y_m), z_1^1, ..., z_{n_m}^m)$$

   Of course in general one would write a commutative diagram for this, or write the general formula $\gamma \circ (\mathrm{id} \times \gamma^m) = \gamma(\gamma \times \mathrm{id}^N)$ where there is a shuffle hidden within. This and subsequent diagrams can be found in [Ric20].

2. The second property is that η forms a unit: we require that both

   $$e \otimes O(n) \to O(1) \otimes O(n) \xrightarrow{\gamma} O(n)$$

   and

   $$O(n) \otimes e^n \to O(n) \otimes O(1)^n \xrightarrow{\gamma} O(n)$$

   commute with the isomorphisms signifying the unitality of e in the monoidal structure.

3. The third property is equidistributivity: if we have a permutation $\sigma \in \Sigma_n$ we get

   $$\begin{array}{ccc} O(n) \otimes O(k_1) \otimes ... \otimes O(k_n) & \xrightarrow{\sigma \otimes \sigma^{-1}} & O(n) \otimes O(k_{\sigma(1)}) \otimes ... \otimes O(k_{\sigma(n)}) \\ \downarrow \gamma & & \downarrow \gamma \\ O(\sum k_i) & \xrightarrow{\sigma(k_1 \dots k_n)} & O(\sum k_i) \end{array}$$

Here $\sigma \otimes \sigma^{-1}$ is defined as the action on $O(n)$ on one factor and permuting the other factors. With $\sigma(k_1, .., k_n)$ we mean the permutation the permutes the blocks of size $k_1, ..., k_n$. Additionally we require

$$\begin{tikzcd}
O(n) \otimes O(k_1) \otimes ... \otimes O(k_n) \arrow[r, "\mathrm{id}\otimes\sigma_1\otimes...\otimes\sigma_n"] \arrow[d, "\gamma"] & O(n) \otimes O(k_1) \otimes ... \otimes O(k_n) \arrow[d, "\gamma"] \\
O(\sum k_i) \arrow[r, "\sigma_1\oplus...\oplus\sigma_n"] & O(\sum k_i)
\end{tikzcd}$$

Here the symbol $\oplus$ denotes concatenation of permutations; an explicit definition is given in section 2.5.

The equidistributivity property signifies that there is always a link between operads and actions of permutations. We will give some examples of operads later, when we can see more of what an operad does. An operad in itself is just a sequence of objects in a category with some operations relating them, and we would like to talk about the structure of the operad in relation with other objects. What we want is comparable to a monoid in a monoidal category $\mathcal{C}$ where we have maps $\mathcal{C} \otimes ... \otimes \mathcal{C} \to \mathcal{C}$, but now we want to involve the operad in these maps.

Definition 2.36. An object X of $\mathcal{C}$ is an **algebra** over a operad O if there are morphisms $\theta_n : O(n) \otimes C^{\otimes n} \to C$. These maps must be associative in a sense, where the following diagram commutes (with $N = \sum k_i$):

$$\begin{tikzcd}
O(n) \otimes O(k_1) \otimes ... \otimes O(k_n) \otimes X^N \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\theta_{k_1}\otimes...\otimes\theta_{k_n}"] & O(N) \otimes X^N \arrow[d, "\theta_N"] \\
O(n) \otimes X^{\otimes n} \arrow[r, "\theta_n"] & X
\end{tikzcd}$$

The unit $O(1)$ must act as a unit (so $e \otimes X \to O(1) \otimes X \xrightarrow{\theta} X$ commutes with $e \otimes X \to X$). Furthermore, we have something akin to equidistributivity, as in that the following diagram commutes:

$$\begin{tikzcd}
O(n) \otimes X_1 \otimes ... \otimes X_n \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\theta"] & O(n) \otimes X_{\sigma(1)} \otimes ... \otimes X_{\sigma(n)} \arrow[d, "\theta"] \\
X \arrow[r, "="] & X
\end{tikzcd}$$

An algebra is also sometimes called an object with an **action** of the operad. We can now discuss some examples of operads, as we now have an idea what we mean by "encoding how far from commutative an operation is".

Example 2.37. The commutativity operad is defined by $O(n) = \{e\}$ the monoidal unit, with all permutations acting trivially and all compositions just resulting in the identity. This can be defined for any strictlty monoidal category. An algebra over the commutative operad is an object with, specialising to $n = 2$, a map $\mu : * \otimes X \otimes X = X \otimes X \to X$ and a map $\eta : e \to O(1) = *$. The fact that μ is invariant under actions of $\sigma \otimes \sigma^{-1}$ means here that we can take the permutation $\sigma = (1\ 2)$ and σ acting trivially on $*$. This means that $\mu \circ (\sigma \cdot) = \mu$; in categories of sets or spaces this becomes that $\mu(x, y) = \mu(y, x)$, so this map is commutative as the name of the operad suggests. The result is thus that we have a strictly commutative monoid. The maps involving higher n work in a similar fashion.

Example 2.38. Let $\mathcal{C}$ be **closed** category; this means that the normal homomorphism sets can be seen as objects of $\mathcal{C}$, and that the homomorphism functor thus obtained is adjoint to the monoidal product. The endomorphism operad of an object C of $\mathcal{C}$ is defined by $O(n) = \mathcal{C}(C^{\otimes n}, C)$. Composition is defined by composition of functions; given $f \otimes f_1 \otimes ... \otimes f_n \in O(n) \otimes O(k_1) \otimes ... \otimes O(k_n)$, we can define the result as, given an product of $\sum k_i$ objects, first applying f_1 to the first k_1, f_2 to the following k_2 etc., and finally on the resulting n objects we apply f. Every object is an algebra of its endomorphism operad by the evaluation map.

Example 2.39. Further coherence conditions can also be defined using operads; for example we could encode abelian groups by using an operad A with $A(0) = \{0\}, A(1) = \{\mathrm{id}, -\}$ where $-$ is the binary minus operation, and $A(2)$ consisting of the four operations sending a, b to $a + b, a - b, b - a, -(a + b)$. We define $\gamma_1(-, -) = \mathrm{id}$. Describing the higher levels of the operad can be done once we have defined the associative operad, but is rather messy. Defining non-abelian groups adds the operations resulting in $b + a$ and $-(b + a)$.

Remark 2.40. We will assume throughout that $O(0)$ is the monoidal unit for all operads O. The idea behind is that the empty product should always be the unit. This then also gives us a map $O(n) \to O(n) \otimes O(0)^n \xrightarrow{\gamma} O(0) = e$ which in the case of spaces gives us what we will call an augmentation. In the original definition by May in [May72] this is explicitly demanded, while other authors have either considered only positive degrees or made the distinction explicit. Suppose we do have an operad with $O(0)$ not the unit and non-empty, and suppose we are working in the category of sets. Then the higher levels of the operad determine a structure on $O(0)$; indeed an element of $O(2)$ then gives a map $O(0) \times O(0) \to O(0)$. If all higher levels have one element as in the commutative operad $O(0)$ becomes a commutative monoid. If all higher levels are the symmetric groups we can give it an associative structure. An algebra A over such an operad will then have a map $\theta_0 : O(0) \to A$, but all elements of the image of this map will act as a unit: $\theta_2(\sigma, \theta_0(\tau), x) = \theta_2(\sigma, \theta_0(\tau), \theta_1(*, x)) = \theta_1(\gamma_2(\sigma, \tau, *), x) = x$ where $*$ is the image of η, using the associative property. Thus allowing non-trivial $O(0)$ just introduces extra units, which we are not interested in.

Example 2.41. The **ring operad** R is given by $R(0) = \{0, 1\}$, $R(1) = \{\mathrm{id}, -\}$ and $R(2)$ generated by $+$ and $\cdot$ (so we also get operations like $a, b \mapsto a - b$ or $(-a) \cdot b$). We use the diagonal map to express the distributive property $a \cdot (b + c) = a \cdot b + a \cdot c$. The ring structure on $R(0)$ is the obvious one.

The following proposition allows us to greatly expand our list of examples of operad algebras. The main application of this will be in defining the Barratt-Eccles operad. It also allows us to freely work with algebras in simplicial sets, as we can directly translate it to topological spaces.

Proposition 2.42. *Let O be an operad in $\mathcal{C}$, and $F : \mathcal{C} \to \mathcal{D}$ a lax symmetric monoidal functor. Then $F(O)$ is an operad in D. If X is an algebra over O, then $F(X)$ is an algebra over $F(O)$.*

Proof. Let O have structure maps γ, η; the structures on $F(O)$ will be denoted γ', η'. We define $\eta' = \eta \circ F$, which is allowed as $F(e_{\mathcal{C}}) = e_{\mathcal{D}}$. Similarly we can define $\gamma'_n = F(\gamma_n) \circ \phi$ where ϕ is the structure map making F lax monoidal. Associativity holds as ϕ is natural and γ is associative. The action of the permutative groups is given by the image under F of the action.

We define θ' on $F(X)$ by $\theta'_n = F(\theta_n) \circ \phi$ in the same way. Associativity follows from the following:

$$\begin{aligned} &\theta'_N \circ (\gamma'_n \otimes' \mathrm{id}) \\ &= F(\theta_N) \circ \phi \circ ((F(\gamma_n) \circ \phi) \otimes' \mathrm{id}) \\ &= F(\theta_N) \circ \phi \circ (F(\gamma_n) \otimes' \mathrm{id}) \circ (\phi \otimes' \mathrm{id}) \\ &= F(\theta_N) \circ F(\gamma_n \otimes \mathrm{id}) \circ \phi \circ (\phi \otimes' \mathrm{id}) \\ &= F(\theta_N) \circ F(\gamma_n \otimes \mathrm{id}) \circ \phi \\ &= F(\theta_N \circ (\gamma_n \otimes \mathrm{id})) \circ \phi \end{aligned}$$

This can then be transformed using the associativity of θ to the desired term in a similar fashion. Here we used the fact that F is a functor and ϕ is natural, and that composition distributes over the monoidal product. It is also to be noted that two copies of ϕ were contracted at some point; if one looks closely this is because the term $F(O)(n) \otimes F(O)(k_1) \otimes ... \otimes F(O)(k_n) \otimes F(X)^N$ is transformed by ϕ in two parts (the operad terms and the X terms) which can be done at once. Equidistributivity also follows: the action of $\sigma \otimes' \sigma^{-1}$ is the image of the action of $\sigma \otimes \sigma^{-1}$ which can be split using the fact that F is lax symmetric. □

Given an operad O in a category of spaces we can make an associated monad. This is usually written

$$M(X) = \coprod_{k=0}^{\infty} O(k) \times_{\Sigma_k} X^{\otimes k}$$

By $\times_{\Sigma_k}$ we mean that we take the normal product $\times$ (which plays the role of the monoidal product here), but we divide out by an equivalence relation such that the action of Σ_k on both sides is carries over to an action on the total. Explicitly, we define $(f, \sigma(x_1, ..., x_k)) \sim (\sigma \cdot f, (x_1, ..., x_k))$ where on the left side we just permute the factors, and on the right we take the action which exists by the definition of an operad. Alternatively, one can say that the product has an action of $\Sigma_k \times \Sigma_k$, and we divide out by elements of the form (σ, σ^{-1}).

For the maps which make this functor a monad, we take as unit $\eta \times \mathrm{id}_X$. As the monoidal unit in the category of spaces is the one-point set we get that $\eta \times \mathrm{id}_X(x) = (*, x)$ which clearly is invariant under the action of Σ_1.

For the composition map we take, where $n = n_1 + ... + n_k$

$$\begin{aligned} M(M(X)) \supset O(k) \times (O(n_1) \times X^{n_1}) \times ... \times (O(n_k) \times X^{n_k}) \\ \to O(k) \times O(n_1) \times ... \times O(n_k) \times X^n \\ \to O(n) \times X^n \subset M(X) \end{aligned}$$

where the first map is just the shuffling of the factors, and the second is the operad map γ. These maps combine into a proper map $M(M(X)) \to M(X)$ because of the properties of the operad map.

Proposition 2.43. *Let O be an operad, and M its associated monad. Then algebras over O are equivalent to algebras over M.*

Proof. A map $f : MX \to X$ is equivalent to a collection of maps $f_k : O(k) \times_{\Sigma_k} X \to X$ (for each of the structure maps i_n of the coproduct, take $f_n = f \circ i_n$). On the other hand, an algebra over O should have a collection of maps $O(k) \times X^k \to X$ which is amongst other things invariant under the action of elements of the form $\sigma \times \sigma^{-1}$, which is exactly what is defined by $\times_{\Sigma_k}$. The other operad algebra properties can also be checked relatively straightforwardly. □

Let X be any object, and the setting as above. Then $M(X)$ is an O-algebra; looking at the computation of the composition map in $M(X)$ we see that this also serves as an algebra structure map. The associated functor $F_O : X \mapsto M(X)$ is also called the **free algebra functor**. The motivation for this statement can be found in Lemma 2.9 of [J P97]. The idea is that it is adjoint to the forgetful map which takes an algebra to its underlying space.

Remark 2.44. The fact that we can express algebras over an operad as algebras over a monad has its benefits: any results we know about algebras of monads now carry over to algebras of operads. For example, in Lemma 9.2 of [SS12] it is shown that a certain category of algebras over an operad has all limits and colimits. The category in question is one we will see in this text, but we will not use the structure exhibited there.

2.5 The Barratt-Eccles operad

Recall the definition of the permutation groups $\Sigma_n = \{\text{bijections } \mathbf{n} \to \mathbf{n}\}$ where $\mathbf{n} = \{1, ..., n\}$. There are maps $- \oplus - : \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, defined by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } i \leq n \\ \tau(i-n)+n & \text{else} \end{cases}$$

Also, given numbers $j_1, ..., j_n$ and $\sigma \in \Sigma_n$, we define $\sigma(j_1, ..., j_n)$ to be the permutation in $\Sigma_{j_1+...+j_n}$ which permutes the blocks of size j_i according to σ.

Definition 2.45. The **associative operad** of sets is the operad defined by $O(n) = \Sigma_n$, and composition maps defined by

$$\gamma(\alpha, \beta_1, ..., \beta_n) = \beta_{\alpha^{-1}(1)} \oplus ... \oplus \beta_{\alpha^{-1}(n)} \cdot \alpha(j_1, ..., j_k)$$

where $\beta_i \in \Sigma_{j_i}$. The permutation groups act on this on the right.

In the article by J. Peter May [May74] where this operad was introduced, the factor $\alpha(j_1, ..., j_n)$ was omitted and later rectified in an erratum in [CLM76], and a simple write-out shows it is indeed necessary. We write a tuple as a tuple as a tuple of blocks within the tuple for convenience, so we write $x_i = (x_i^1, ..., x_i^{j_i})$; the tuple $(x_1, ..., x_n)$ is thus not an n-tuple but a concatenation of n tuples. We get, with the permutations as in the definition,

$$\begin{aligned} &\gamma(\alpha, \beta_1, ..., \beta_n)(x_1, ..., x_n) \\ &= \beta_{\alpha^{-1}(1)} \oplus ... \oplus \beta_{\alpha^{-1}(n)}(x_{\alpha^{-1}(1)}, ..., x_{\alpha^{-1}(n)}) \\ &= (\beta_{\alpha^{-1}(1)} \cdot x^1_{\alpha^{-1}(1)}, ..., \beta_{\alpha^{-1}(1)} \cdot x^{\alpha^{-1}(1)}_{\alpha^{-1}(1)}, ..., \beta_{\alpha^{-1}(n)} \cdot x^1_{\alpha^{-1}(n)} ..., \beta_{\alpha^{-1}(n)} \cdot x^{\alpha^{-1}(n)}_{\alpha^{-1}(n)}) \end{aligned}$$

We see that the term $\alpha(j_1, ..., j_n)$ is necessary to align the permutations with the proper arguments, as otherwise the permutation $\beta_{\alpha^{-1}(1)}$ would not be acting on a predetermined block of length $j_{\alpha^{-1}(1)}$ but rather on the first $j_{\alpha^{-1}(1)}$ arguments which may not form a block. While this is not a priori a problem, we will see that equivariance is not satisfied without this condition.

Example 2.46. Before we fully check all properties to verify this is an operad, let us look at what it means to be an algebra over this operad: suppose X is such a set. Then we have a map $\theta_n : \Sigma_n \times X^n \to X$ for all n. Equivariance forces the definition of this map: $\theta_n(\sigma, (x_1, ..., x_n)) = \theta_n(\mathrm{id}, (x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)})$. We can then write all expressions in terms of $\cdot = \theta_2(\mathrm{id}, -)$. This multiplication map is associative, as

$$x \cdot (y \cdot z) = \theta_3(\gamma_2(\mathrm{id}_2, \mathrm{id}_1, \mathrm{id}_2), (x, y, z)) = \theta_3(\gamma(\mathrm{id}_2, \mathrm{id}_2, \mathrm{id}_1), (x, y, z)) = (x \cdot y) \cdot z$$

The definition of our associative operad thus gives sets with associative operations and a unit as algebras, which we know as monoids.

Proposition 2.47. *The associative operad forms an operad*

Proof. We will not check associativity. The unit map obviously acts as a unit: $O(1) = \Sigma_1 = \{\mathrm{id}\}$, and $\gamma(\mathrm{id}, \sigma) = \sigma$. For the first equivariance condition, we need that $\gamma \circ (\sigma \times \sigma^{-1}) = \sigma(j_{\sigma(1)}, ..., j_{\sigma(n)}) \circ \gamma$. We compute, again on $(\alpha, \beta_1, ..., \beta_n)$:

$$\sigma \times \sigma^{-1}(\alpha, \beta_1, ..., \beta_n) = (\alpha \cdot \sigma, \beta_{\sigma(1)}, ..., \beta_{\sigma(n)})$$
$$\gamma(\alpha \cdot \sigma, \beta_{\sigma(1)}, ..., \beta_{\sigma(n)}) = (\beta_{\alpha^{-1}(1)} \oplus ... \oplus \beta_{\alpha^{-1}(n)}) \cdot (\alpha \cdot \sigma)(j_{\sigma(1)}, ..., j_{\sigma(n)})$$

Here we use that permutations act on the right, and for the indexing of the β-terms we use that $\sigma(\alpha\sigma)^{-1} = \alpha^{-1}$. On the other hand, we compute:

$$\sigma(j_{\sigma(1)}, ..., j_{\sigma(n)})\gamma(\alpha, \beta_1, ..., \beta_n)$$
$$= (\beta_{\alpha^{-1}(1)} \oplus ... \oplus \beta_{\alpha^{-1}(n)}) \cdot \alpha(j_1, ..., j_n) \cdot \sigma(j_{\sigma(1)}, ..., j_{\sigma(n)})$$
$$= (\beta_{\alpha^{-1}(1)} \oplus ... \oplus \beta_{\alpha^{-1}(n)}) \cdot (\alpha \cdot \sigma)(j_{\sigma(1)}, ..., j_{\sigma(n)})$$

The two results are equal, so we see that this condition is indeed satisfied. □

Since we have defined an operad in the category of sets, we could just take the discrete variants of this set to get an operad in small categories or spaces. This gives sensible results as it allows us to define associative monoids in these categories, but we want more; we want to get a concept of algebras which are associative but only commutative up to some notion of coherence. Such a structure in the category of (small) categories is a permutative category. To construct what the operad O which permutative categories are algebras of looks like, let us look in degree 2: there are two 2-ary operations, namely μ and $\mu \circ \tau$ with μ the multiplication and τ the symmetry isomorphism. On the level of the operad this must mean that there is a map $\circ\tau$ between the two elements of $O(2)$ which is an isomorphism. The general pattern for this operad can be easily described using the following concept:

Definition 2.48. Given a group G, we can form its **translation category** denoted $\tilde{G}$ where the objects are the elements of G, and $\tilde{G}(g, h)$ has one element which we think of (or denote) as hg^{-1}.

We can use the translation category construction to lift our associative operad to an operad on small categories. The only new requirement is that now the composition map is not only a map of objects (so, a map of sets as our category is small), but a functor, so we also need to define it on morphisms.

Definition 2.49. The **Barratt-Eccles operad** is the collection $\tilde{\Sigma}_n$ in the monoidal category $(\mathrm{cat}, \times, 1)$. The composition map on objects is the same as for the associative operad. The composition map on morphisms is given by

$$\gamma(\sigma\alpha^{-1}, \tau_1\beta_1^{-1}, ..., \tau_n\beta_n^{-1}) = \gamma(\sigma, \tau_1, ..., \tau_n)\gamma(\alpha, \beta_1, ..., \beta_n)^{-1}$$

There is only one morphism between any two objects, so by definition this defines the morphism $\gamma(\alpha, \beta_1, ..., \beta_n) \to \gamma(\sigma, \tau_1, ..., \tau_n)$. It is not the same as applying γ to the objects $\sigma\alpha^{-1}, \tau_1\beta_1^{-1}, ..., \tau_n\beta_n^{-1}$ seen as objects; this is almost the case, but we can see that, attempting to write this in the same format as the definition of γ on objects, we get terms of the form $\tau_{\alpha\sigma^{-1}(i)}\beta^{-1}_{\alpha\sigma^{-1}(i)}$ which do not fit in what we have defined above when we expand it:

$$\begin{aligned} &\gamma(\sigma\alpha^{-1}, \tau_1\beta_1^{-1}, .., \tau_n\beta_n^{-1}) \\ = \tau_{\sigma^{-1}(1)}\beta^{-1}_{\alpha^{-1}(1)} \oplus ... \oplus \tau_{\sigma^{-1}(n)}\beta^{-1}_{\alpha^{-1}(n)} &\cdot (\sigma\alpha^{-1})(j_{\alpha^{-1}(1)}, ..., j_{\alpha^{-1}(n)}) \end{aligned}$$

We will check the statement that this is a natural map by applying the left map to a sequence of elements

$$(x[1,1], ..., x[1, j_1], ..., x[n,1], ..., x[n, j_n])$$

which we will abbreviate by the product notation $\prod_i (x[i,1], ...)$ as the rest of the indices can be deduced from this. The reason that we have chosen this non-conventional notation and not super- and subscripts is to improve legibility, not to suggest any function-like property of the x. Applying to such a sequence does not make any sense strictly speaking; the $\alpha, \beta, ...$ are objects in a category without any meaning of action, but we can work with the permutations they represent without any loss of correctness. We will also as a shorthand write $(\tau\beta^{-1})_i$ for $\tau_i\beta_i^{-1}$. We compute:

$$\begin{aligned}
&\gamma(\sigma\alpha^{-1}, \tau_1\beta_1^{-1}, .., \tau_n\beta_n^{-1})\gamma(\alpha, \beta_1, .., \beta_n)(\prod_i (x[i,1], ...)) \\
&= \gamma(\sigma\alpha^{-1}, \tau_1\beta_1^{-1}, .., \tau_n\beta_n^{-1}) \bigoplus_i \beta_{\alpha^{-1}(i)} \cdot (\alpha(j_1, ..., j_n)(\prod_i (x[i,1], ...)) \\
&= \gamma(\sigma\alpha^{-1}, \tau_1\beta_1^{-1}, .., \tau_n\beta_n^{-1}) \bigoplus_i \beta_{\alpha^{-1}(i)}(\prod_i (x[\alpha^{-1}(i), 1], ...)) \\
&= \gamma(\sigma\alpha^{-1}, \tau_1\beta_1^{-1}, .., \tau_n\beta_n^{-1})(\prod_i (x[\alpha^{-1}(i), \beta^{-1}_{\alpha^{-1}(i)}(1)], ...)) \\
&= \bigoplus_i \tau_{\sigma^{-1}(1)}\beta^{-1}_{\alpha^{-1}(1)} \cdot (\sigma\alpha^{-1})(j_{\alpha^{-1}(1)}, ..., j_{\alpha^{-1}(n)})(\prod_i (x[\alpha^{-1}(i), \beta^{-1}_{\alpha^{-1}(i)}(1)], ...)) \\
&= \bigoplus_i \tau_{\sigma^{-1}(1)}\beta^{-1}_{\alpha^{-1}(1)}(\prod_i (x[\alpha^{1}(\alpha\sigma^{-1}(i)), \beta^{-1}_{\alpha^{-1}(i)}(1)], ...)) \\
&= \prod_i (x[\sigma^{-1}(i), \beta^{-1}_{\alpha^{-1}(i)}(\beta_{\alpha^{-1}(i)}\tau^{-1}_{\sigma^{-1}(i)}(1))], ...) \\
&= \prod_i (x[\sigma^{-1}(i), \tau^{-1}_{\sigma^{-1}(i)}(1)], ...) \\
&= \gamma(\sigma, \tau_1, .., \tau_n)(\prod_i (x[i,1], ...))
\end{aligned}$$

The fact that operads are preserved under lax monoidal functors gives us various new operads for free. The first is an operad in simplicial sets given by $N\tilde{\Sigma}_n$. From this we can get to an important milestone: the operad $B\tilde{\Sigma}_n$ on topological spaces. These two form what is more commonly known as the Barratt-Eccles operad.

In practice it is easier to work with the category version or the simplicial version; the algebraic properties are more clearly expressed in the categorical or simplicial context. As an example, $B\tilde{\Sigma}_1$ is a point. However, for $B\tilde{\Sigma}_2$ we get two points, with two lines (1-cells) for the isomorphism between the two objects, then two 2-cells etc. This is the example from Example 2.19, which yields the infinite-dimensional sphere which is well-understood but which contains a lot more data than the simplicial set or category underlying it. For $B\tilde{\Sigma}_3$ we get six points with $6(6-1)/2 * 2 = 30$ lines. This further escalates as between any two points we are building a copy of S^∞ but there are also faces and higher simplices connecting these. These spaces do have some favourable properties (they are contractible and have a free action by Σ_n as we will see) but in practice we will not use these in this text.

2.6 The action of the Barratt-Eccles operad

We can now define the more useful property of the Barratt-Eccles operad on certain interesting topological spaces:

Theorem 2.50. *A permutative category $\mathcal{C}$ carries an action of the Barratt-Eccles operad of categories.*

Proof. We define $\theta : \tilde{\Sigma}_n \otimes \mathcal{C}^n \to \mathcal{C}$ by $\theta(\alpha, C_1, ..., C_n) = C_{\alpha^{-1}(1)} \otimes ... \otimes C_{\alpha^{-1}(n)}$. We check: $\theta(\tau \otimes \tau^{-1}(\sigma, C_1, ..., C_n)) = \theta(\tau \cdot \sigma, C_{\tau(1)}, .., C_{\tau(n)}) = C_{\sigma^{-1}\tau^{-1}(\tau(1))} \otimes ... \otimes C_{\sigma^{-1}\tau^{-1}(\tau(n))} = C_{\sigma^{-1}(1)} \otimes ... \otimes C_{\sigma^{-1}(n)}$ □

Corollary 2.51. *Let $\mathcal{C}$ be a permutative category. Then $N\mathcal{C}$ and $B\mathcal{C}$ are algebras of the Barratt-Eccles operad in simplicial sets or spaces*

Proof. This follows by the statement of Proposition 2.42 and the fact that both the nerve and the geometric realization functors are lax monoidal. □

The most notable occurrence of the Barratt-Eccles operad is the following:

Theorem 2.52. *Infinite loop spaces, i.e. spaces X for which there are homotopy equivalences $X \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq ... \simeq \Omega^n X_n \simeq ...$ for all n, are algebras of the Barratt-Eccles operad.*

The converse also holds up to weak equivalence. This is known as the recognition principle and is due to J.P. May; a precise statement can be found in theorem 1.3 of [May72].

2.7 E_∞-operads

In the literature one often sees the notion of an E_∞-operad. This is a general term for operads modeling commutativity relations holding up to weak equivalence.

Definition 2.53. Let O be an operad of spaces. Then O is an E_∞-operad if all $O(n)$ are contractible have a free Σ_n-action.

Recall that an action G on a set X is free if $g \cdot x = h \cdot x$ implies that $g = h$, or equivalently $g \cdot x = x$ implies that g is the unit. Note that this is quite strict and rules out many spaces we intuitively think of as symmetric; for a space this means that there is no point or line or plane of symmetry. Giving a space with a free action of a symmetric group even of order 2 seems quite hard in the topological setting, but as a category we see that $\tilde{\Sigma}_2$ has such a free action; we can then use the classifying space to get a topological space with this property.

Proposition 2.54. *The Barratt-Eccles operad is an E_∞-operad.*

Proof. The action is given on the level of simplicial sets by

$$\tau \cdot (\sigma_1 \to ... \to \sigma_n) = \tau\sigma_1 \to ... \to \tau\sigma_n$$

with the unique maps between them. Since every σ induces an automorphism of the permutation group we get that $\tau \cdot \sigma = \tau \cdot \sigma' \implies \sigma = \sigma'$.

For the contractability, we notice that this is an elaboration of the example at Example 2.19, as the category studied there is exactly $\tilde{\Sigma}_2$. Since every object is initial we also apply Proposition 2.23. In general, the homotopy between the inclusion of a point and the identity will be

$$H_n(0, (\sigma_1, ..., \sigma_n) = (\sigma_0, ..., \sigma_n),$$
$$H_n(n+1, (\sigma_0, ..., \sigma_n)) = (\mathrm{id}, ..., \mathrm{id}),$$
$$H_n(i, (\sigma_0, ..., \sigma_n)) = (\mathrm{id}, ..., \mathrm{id}, \sigma_{i+1}\sigma_i ... \sigma_0, \sigma_{i+2}, ..., \sigma_n)$$

This gives a simplicial homotopy as one can check, and we are done. □

Definition 2.55. Let O, P be operads, and f a family of morphisms $f_n : O(n) \to P(n)$. Then f is a **morphism of operads** if $f_n(\gamma_O(x, y_1, ..., y_k) = \gamma_P(f_k(x), f_{l_1}(y_1), ..., f_{l_k}(y_k))$ for $x \in O(k), y_i \in O(l_i)$, and $f_1 \circ \eta_O = \eta_P$.

Examples:

- A morphism from an operad O to the endomorphism operad on a space gives that space the structure of an O-algebra.
- The path component functor π_0 where the image is now seen as a discrete topological space. This means that if O is an operad of topological spaces, then the $\pi_0(O(n))$ also form an operad of spaces and the maps $O(n) \to \pi_0(O(n))$ are compatible with the operad structure.

This gives the class of operads within a monoidal category the structure of a category itself. The commutative operad defines a terminal object in the category of operads of sets (or any monoidal category where the terminal category is the monoidal unit), and the initial object is given by the operad with

$O(0) = \{0\}, O(1) = \{\mathrm{id}\}, O(n) = \emptyset$ for $n > 1$. Remember that we silently assumed $O(0) = e$; if we do not there is obviously no initial object.

The term E in E_∞ comes from the idea that "everything commutes up to homotopy". There is another variant, called A_n or A_∞ where "everything is associative up to homotopy". The degree in between A_∞ and E_∞ are the E_n-operads, where A_∞ is equivalent E_1. These operads encode operations which are commutative up to level n. This notion is generally of lesser interest, as in some important categories (such as the category of sets or abelian groups) being E_2 gives all the higher commutativity as well by using the Eckmann-Hilton argument. In topological spaces these do become more interesting: n-fold loop spaces are E_n-algebras. The converse also holds up to weak equivalence. This also works when we let n go to infinity, and we get the aformentioned result that E_∞-algebras of topological spaces are infinite deloopings.

We will not give a precise definition of A_n of E_n-operads, as this is not in the scope of this text.

From this point on we will call all algebras of the Barratt-Eccles operad **E_∞-spaces**. The reason is that for any other E_∞-operad O the category of O-algebras is Quillen equivalent to that of the Barratt-Eccles algebras. This statement of course relies on the fact that there is a model structure on operad algebras, and also on a model structure for the category of operads. The theorem would require much more development of the homotopy theory surrounding operads which we will not do in this text. A reference for this fact is [BM03]; theorem 4.4 and the consequent remark 4.6 give the exact statement. The definition of an E_∞-operad is that it has some cofibrancy property and that there is an operad morphism with some specific properties to the commutative operad. This also gives the parallel to the A_∞-operads mentioned above; these are operads with the same properties but the morphism in this case is to the associative operad.

3 $\mathcal{I}$-spaces

In this section we will explore the concept of $\mathcal{I}$-spaces. These are some form of coherently indexed families indexed by the category $\mathcal{I}$ which we will define. The interesting fact is that the category of $\mathcal{I}$-spaces carries a monoidal product derived from the product in the category $\mathcal{I}$, and that commutative monoids in this structure are somewhat less strict than being a space with a commutative operation. They will tie into the Barratt-Eccles operad we defined earlier; the correspondence is that commutative $\mathcal{I}$-spaces are equivalent to E_∞-spaces up to weak equivalence.

3.1 Definitions

Definition 3.1. The category $\mathcal{I}$ has the set $\mathbb{N}$ on objects, denoted in bold as in $\mathbf{n}$. We define $\mathcal{I}(\mathbf{m}, \mathbf{n})$ to be the injections $\{1, ..., m\} \to \{1, ..., n\}$ (so $\mathbf{n}$ corresponds to $\{1, ..., n\}$ while $\mathbf{0}$ corresponds to the empty set). Let $\mathcal{S}$ be a category of spaces. An **$\mathcal{I}$-space** X is a functor $\mathcal{I} \to \mathcal{S}$. The category of $\mathcal{I}$-spaces is denoted $\mathcal{S}^{\mathcal{I}}$, with morphisms being the natural transformations between these functors.

In the above, a "category of spaces" can be different depending on the context; it can be useful to work with either simplicial sets or topological spaces depending on the situation. The category $\mathcal{I}$ is not the only setting in which one might be interested. The structures obtained by looking at functors from a small monoidal category to a category of spaces have a name: they are called **diagram spaces**. Later on we will broaden our scope to $\mathcal{K}$-spaces where $\mathcal{K}$ can be any permutative category.

Let us spell out what it means to be an $\mathcal{I}$-space. To start with, we need a sequence of objects $X(\mathbf{n})$. Next to that we need, for every injection $f : \mathbf{n} \to \mathbf{m}$ a map $X(f) : X(\mathbf{n}) \to X(\mathbf{m})$ which should of course behave well under composition. When the context is clear this map will also be denoted f_* When taking $n = m$, we see that we get a group homomorphism from Σ_n to $\mathrm{Aut}(X(\mathbf{n}))$, so an action of the permutation group.

A morphism of $\mathcal{I}$-spaces is a natural transformation of functors, so $F : X \to Y$ is an $\mathcal{I}$-indexed collection of morphisms of spaces such that the following commutes:

$$\begin{tikzcd} X(\mathbf{n}) \arrow[r, "X(f)"] \arrow[d, "F(\mathbf{n})"] & X(\mathbf{m}) \arrow[d, "F(\mathbf{m})"] \\ Y(\mathbf{n}) \arrow[r, "Y(f)"] & Y(\mathbf{m}) \end{tikzcd}$$

Examples:

- For any space S we can form the constant $\mathcal{I}$-space $\mathbf{n} \mapsto S$ where all functions are mapped to the identity. When the space is initial this is the initial object, and vice verse for the terminal object.

- The functor $\mathbf{n} \mapsto \{1, ..., n\}$ as discrete spaces. The injections are mapped to themselves.

- For any based space (X, x_0), the functor $\mathbf{n} \mapsto X^n$, where X^n is the n-fold product of X. For injections $f : \mathbf{n} \to \mathbf{m}$ we get $(f_*(x_1, ..., x_n))_i = x_j$ if $f(j) = i$ or $j = 0$ whenever i is not in the image of f, resulting in the basepoint x_0.

- The matrix groups $\mathbf{n} \mapsto O(n)$ over any field. Recall that a permutation matrix of a permutation $\sigma \in \Sigma_n$ is the matrix obtained by permuting the columns of the identity matrix of dimension n. Conjugation by a permutation matrix gives an action of the Σ_n on $O(n)$. We also have for every n a distinguished map $\alpha : \mathbf{n} \to \mathbf{n+1}$ which misses the element $n+1$; this map in $\mathcal{I}$ we associate to the map of matrices

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

where we put one extra 1 on the diagonal and fill with zeroes. We can write all maps $j \in \mathcal{I}(\mathbf{n}, \mathbf{m})$ as a composition $j : \sigma \circ \alpha^r$ for some $\sigma \in \Sigma_m$, and thus we can define the map $j^* : O(n) \to O(m)$ by first applying the map associated to α a total or r times and then conjugation by the permutation matrix of σ. This gives a well-defined functor, as we can write

$$\sigma \circ \alpha^r \circ \tau \circ \alpha^p = \sigma \circ (\tau \oplus \mathrm{id}_r) \circ \alpha^{r+p}$$

We also see that $(\tau \oplus \mathrm{id}_r)^{-1} = \tau^{-1} \oplus \mathrm{id}_r$, and that the permutation matrix of $\tau \oplus \mathrm{id}_r$ is the permutation matrix of τ with 1 on the diagonal for all higher rows and columns. One can then write out that this means that this indeed commutes with composition. It is clear that the permutation matrices are invertible and orthogonal. The operation associated to α also clearly makes orthogonal matrices out of orthogonal matrices. In conclusion this indeed gives a functor.

The above is carried out while viewing $O(n)$ as a category with one object and arrows for all the elements.

We could have done the same for the groups $GL(n)$ and $U(n)$ or their special counterparts.

- The spheres S^n, with the small change that we set $S(n) = S^{n-1}, S(0) = \emptyset$. An injective map $f : \mathbf{n} \to \mathbf{m}$ is taken to the inclusion

$$(x_1, ..., x_n) \mapsto (x_{f^{-1}(1)}, ..., x_{f^{-1}(m)})$$

where $x_{f^{-1}(i)} = 0$ if there is no j with $f(j) = i$; intuitively this is the same as applying the function f to the set of n coordinates of S^{n-1}. This takes an element of $S(n)$ to an element of $S(m)$ as the sum of all the squares of indices is still 1. Functoriality follows by intuitively seeing that if $f(j) = j$ and $g(j) = k$ that then $S(g)((S(f)(x)) = S(g \circ f)(x)$ as the k'th coordinate in both will be x_i and this works for all i, j, k. This means that one formally gets that

$$S(g)(x_{f^{-1}(1)}, ..., x_{f^{-1}(m)}) = (x_{f^{-1}(g^{-1}(1))}, ..., x_{f^{-1}(g^{-1}(m))})$$

which is somewhat counter-intuitive but does correspond to the interpretation of injective functions applied to the set of coordinates when one writes it out in an example.

The category $\mathcal{I}$ has a monoidal structure denoted $\sqcup$. We define $\mathbf{m} \sqcup \mathbf{n}$ to be the object associated to $m + n$. If $f : \mathbf{m}_1 \to \mathbf{n}_1, g : \mathbf{m}_2 \to \mathbf{n}_2$ we define $f \sqcup g : \mathbf{m}_1 \sqcup \mathbf{m}_2 \to \mathbf{n}_1 \sqcup \mathbf{n}_2$ to be the injection which acts by f on the first m_1 and by g on the last m_2. The monoidal unit is $\mathbf{0}$. The associativity conditions are straightforward. The monoidal structure is symmetric by $\tau_{m,n} : \mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$ being the permutation which swaps the two blocks. This is not a strict symmetry.

We want to somehow extend this monoidal structure from $\mathcal{I}$ to $\mathcal{I}$-spaces. We could of course put $(X \times Y)(\mathbf{n}) = X(\mathbf{n}) \times Y(\mathbf{n})$, but would not give us very interesting results in our context: all objects would be monoids by using the projection functions, and we would not be able to relate to $\sqcup$ whatsoever. What we want is that monoids come with maps $X(\mathbf{n}) \times X(\mathbf{m}) \to X(\mathbf{n} \sqcup \mathbf{m})$ for all n, m.

We first define, for $\mathcal{I}$-spaces X, Y, $(X \times Y)(\mathbf{n}, \mathbf{m}) = X(\mathbf{n}) \times Y(\mathbf{m})$. This is an example of an $\mathcal{I}^2$-space. What we have now is a diagram of the form:

$$\begin{tikzcd} \mathcal{I} \times \mathcal{I} \arrow[r, "X \times Y"] \arrow[d, "\sqcup"] & \mathcal{S} \\ \mathcal{I} \end{tikzcd}$$

What we want now is to extend this to get a third map $\mathcal{I} \to \mathcal{S}$ which "approximates" a completion of this diagram; to make this a unique construction, we ask that it is the best approximation, in that it is closest to making the diagram commute. This is the concept of a left Kan extension:

Definition 3.2. Given functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{C}$, we define the **left Kan extension** of F by G to be a functor $\mathrm{Lan}_G F : \mathcal{C} \to \mathcal{B}$ with a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_G F \circ G$ which forms an initial pair with this property.

Explicitly, given the data as in the definition, if the left Kan extension exists, we get a diagram like:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{G}"'] \arrow[dr, phantom, "\Downarrow \eta"] & \mathcal{B} \\ \mathcal{C} \arrow[ur] \arrow[ur, bend right] \arrow[ur, phantom, bend right=20, "\Searrow \epsilon"] & \end{tikzcd}$$

where $\eta : F \Rightarrow \mathrm{Lan}_G F$. If there is such another arrow $L : \mathcal{C} \to \mathcal{B}$ with a natural transformation $\delta : L \circ G \Rightarrow F$, we get a unique natural transformation $\epsilon : \mathrm{Lan}_G F \Rightarrow L$ as above such that $\delta = \epsilon \circ \eta$. Note that in the diagram the functors may not commute: the Kan extension is a "closest approximation", and a directly commuting functor may not exist. The natural transformations do commute. If there is a way to make the diagram commute then this forms the Kan extension; the natural transformation η in that case is the identity which we can precompose with any other way to approximate the diagram to give a factorisation.

Left Kan extensions can be calculated pointwise under favourable conditions. For a functor $G : \mathcal{A} \to \mathcal{C}$ and an object $C \in \mathcal{C}$, we define the category $G \downarrow C$ to have as objects pairs (A, f) with $A \in \mathcal{A}$ and $f : G(A) \to C$, with morphisms being the $g : A \to A'$ such that the following diagram commutes:

$$\begin{tikzcd}
G(A) \arrow[rr, "\mathrm{G(g)}"] \arrow[dr, "f"'] & & G(A') \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

Lemma 3.3. *If the following colimit exists for all $D \in \mathcal{C}$ we may define the left Kan extension by*

$$(\mathrm{Lan}_G F)(D) = \operatorname*{colim}_{G\downarrow D} F \circ U$$

where U is the forgetful functor $G \downarrow D \to \mathcal{A}$

Lemma 3.4. *We have that* Lan_G *is adjoint to precomposition with G, so we get the following property:*

$$\mathcal{B}^{\mathcal{C}}(\mathrm{Lan}_G F, L) = \mathcal{B}^{\mathcal{A}}(F, L \circ G)$$

The proof of both these lemmas can be found in chapter IV.1 of [Ric20]. We will not go into full detail in this subject, and leave details such as the fact that Kan extensions are stable under natural isomorphisms to the reader.

We can now define the monoidal product $X \boxtimes Y$ in the category of $\mathcal{I}$-spaces to be the left Kan extension of $X \times Y$ along $\sqcup$; the diagram takes the shape

$$\begin{tikzcd}
\mathcal{I} \times \mathcal{I} \arrow[r, "X\times Y"] \arrow[d, "\sqcup"] & \mathcal{S} \\
\mathcal{I} \arrow[ur, "X\boxtimes Y"'] &
\end{tikzcd}$$

Proposition 3.5. *The category of $\mathcal{I}$-spaces is a symmetric monoidal category with this structure.*

Proof. For associativity we can draw the diagram

$$\begin{tikzcd}
\mathcal{I}^3 \arrow[r, "(X\times Y)\times \mathrm{id}"] \arrow[d, "\sqcup\times\mathrm{id}"] & \mathcal{S}\times\mathcal{I} \arrow[r, "\mathrm{id}\times Z"] & \mathcal{S} \\
\mathcal{I}^2 \arrow[ur, "(X\boxtimes Y)\times\mathrm{id}"'] \arrow[d, "\sqcup"] & & \\
\mathcal{I} \arrow[uurr, bend right, "(X\boxtimes Y)\boxtimes Z"'] & &
\end{tikzcd}$$

Here $(X \boxtimes Y) \boxtimes Z$ is the Kan extension of $(\mathrm{id} \times Z) \circ ((X \boxtimes Y) \times \mathrm{id})$ along $\sqcup$. If we let η be the natural transformation belonging to the triangle defining $(X \boxtimes Y) \times \mathrm{id}$, and ϵ the one for $(X \boxtimes Y) \boxtimes Z$, we can compose. Note that

$$\eta : (X \times Y) \times \mathrm{id} \Rightarrow ((X \boxtimes Y) \times \mathrm{id}) \circ (\sqcup \times \mathrm{id})$$

so by composing we get the natural transformation

$$(\mathrm{id} \times Z) \circ \eta : (X \times Y) \times Z \Rightarrow (X \boxtimes Y) \times Z \circ (\sqcup \times \mathrm{id})$$

The result is that we get

$$\epsilon \circ (\mathrm{id} \times Z) \circ \eta : (X \times Y) \times Z \Rightarrow ((X \boxtimes Y) \boxtimes Z) \circ \sqcup \circ (\sqcup \times \mathrm{id})$$

which exhibits the outer triangle as a left Kan extension, so $(X \boxtimes Y) \boxtimes Z$ as the extension of $(X \times Y) \times Z$ along $\sqcup \circ (\sqcup \times \mathrm{id})$. These two functors can by associativity be seen as naturally isomorphic to $X \times (Y \times Z)$ and $\sqcup \times (\mathrm{id} \times \sqcup)$. We can form the same diagram to exhibit $X \boxtimes (Y \boxtimes Z)$ as the Kan extension of the latter two, so we must have that the two extensions are naturally isomorphic.

The unit is the $\mathcal{I}$-space which is usually thought of as $\mathcal{I}(\mathbf{0}, -)$, but any $\mathcal{I}$-space e where $e(\mathbf{n})$ has one element for each $\mathbf{n}$ and all maps are sent to the trivial map will do. To see this, we get $(X \boxtimes e)(\mathbf{n}) = \mathrm{colim}_{\sqcup \downarrow \mathbf{n}} X \times e = \mathrm{colim}_{\sqcup \downarrow \mathbf{n}} X = X(\mathbf{n})$ as the colimit runs over all the $X(\mathbf{m}), m \leq n$ with the maps from the injections; $X(\mathbf{n})$ is clearly a colimiting cocone for this diagram.

For commutativity we define the map τ for all $\mathbf{n}, \mathbf{m}$ by the swapping map $s : X(\mathbf{n}) \times Y(\mathbf{m}) \to Y(\mathbf{m}) \times X(\mathbf{n})$. More explicitly, when we also involve the block swapping map $\chi : \mathbf{n} \sqcup \mathbf{m} \to \mathbf{m} \sqcup \mathbf{n}$, we get the description

$$\tau(X \boxtimes Y)(\mathbf{n}) = \operatorname*{colim}_{\sqcup \circ \chi \downarrow \mathbf{n}} s \circ (X \times Y) = \operatorname*{colim}_{\sqcup \downarrow \mathbf{n}} Y \times X = (Y \boxtimes X)(\mathbf{n})$$

Here we used the fact that $\circ\chi$ and the swapping map are isomorphisms and thus do not change the colimit. Clearly $\tau \circ \tau = \mathrm{id}$. Other (larger) diagrams are left to the imagination of the reader.

$\square$

Example 3.6. For constant $\mathcal{I}$-spaces written C_S, we see that

$$(C_S \boxtimes C_{S'})(\mathbf{n}) = \operatorname*{colim}_{\sqcup \downarrow \mathbf{n}} S \times S' = S \times S' = C_{S \times S'}(\mathbf{n})$$

Example 3.7. In general there is no clear picture of what a product of $\mathcal{I}$-spaces looks like. Take for example $O \boxtimes O$; when we have developed the technique of coends in the next section we can write $(O \boxtimes O)(\mathbf{n}) = \coprod_{\mathbf{m}_1, \mathbf{m}_2} \mathcal{I}(\mathbf{m}_1 \sqcup \mathbf{m}_2, \mathbf{n}) \times O(\mathbf{m}_1) \times O(\mathbf{m}_2)/\sim$. The equivalence relation is given by $(h \circ (f \sqcup g), A, B) \sim (h, f^*A, f^*B)$. This can be interpreted as meaning that it is all the possible pairs of matrices with a given way to place them into an $n \times n$-matrix, with the understanding that shuffling one of the matrices is the same as shuffling the way they are put in.

The fact that we have an adjunction as in Lemma 3.4 means that we can at least work with maps out of $X \boxtimes Y$; we get that $\mathcal{S}^{\mathcal{I}}(X \boxtimes Y, Z) \cong \mathcal{S}^{\mathcal{I}^2}(X \times Y, Z \circ \sqcup)$. This means that a family of maps $(X \boxtimes Y)(\mathbf{n}) \to Z(\mathbf{n})$ is equivalent to a family of maps $X(\mathbf{m}) \times Y(\mathbf{n}) \to Z(\mathbf{m} \sqcup \mathbf{n})$.

Let us analyse what it means to be a monoid in this category; suppose X is such a monoid. Then we have a map $X \boxtimes X \to X$ of $\mathcal{I}$-spaces, which by the adjunction described above corresponds to a map $X \times X \to X \circ \sqcup$ of $\mathcal{I}^2$-spaces. This means that for every n, m we get a map $X(\mathbf{n}) \times X(\mathbf{m}) \to X(\mathbf{n} \sqcup \mathbf{m})$.

If we have a commutative monoid, we get an isomorphism $\tau : X \boxtimes X \to X \boxtimes X$. If τ also denotes the corresponding map under the adjunction, we get a square

$$\begin{tikzcd} X(\mathbf{n}) \times X(\mathbf{m}) \arrow[r] \arrow[d, "\tau"] & X(\mathbf{m} \sqcup \mathbf{n}) \arrow[d, "X(\chi)"] \\ X(\mathbf{m}) \times X(\mathbf{n}) \arrow[r] & X(\mathbf{n} \sqcup \mathbf{m}) \end{tikzcd}$$

Here χ denotes the n, m-shuffle.
Examples:

- Constant $\mathcal{I}$-spaces of monoids can be given a monoid structure using the normal multiplication.
- The ortogonal groups $\mathbf{n} \mapsto O(n)$. The multiplication is given by

  $$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

  where the 0's denote blocks of zeroes of appropriate size. We view $O(0)$ as consisting of only the empty matrix, which acts as a unit. It is to be noted that this structure on the orthogonal groups has nothing to do with their multiplicative structure
- The product spaces $\mathbf{n} \mapsto X^n$ for X based space; the maps $X^n \times X^m \to X^{n+m}$ are the obvious ones. As explained before we need the designated basepoint to define the actions of the injections.
- The sphere $\mathcal{I}$-space we defined by $S(\mathbf{n}) = S^{n-1}$ does not define a monoid. When one tries to write down the obvious monoidal product this turns out to be not associative by the fact that the squares of indices must add up to 1.

3.2 Coends and ends

Coends are a type of colimit that appears frequently in the theory. The use for them stems from the fact that there are some very useful theorems regarding the manipulation of coends and their covariant counterpart, ends. Coends over some of the most elementary functors yield very convenient results: for example the set of natural transformations between functors F and G is just the end over the homomorphism-set functor $\mathcal{C}(F(-), G(-))$. Variations of this fact in combination with for example the Yoneda lemma yield very useful identities which we can utilise to talk about abstract constructions such as products of $\mathcal{I}$-spaces and the homotopy colimits we will see later in a more unified way.

Most of the material in this section is taken from [Lor15]. [Ric20] also discusses (co)ends in chapter IV.4. The article [HV92] also defines coends, much in the same (non-standard) way we do here.

Definition 3.8. Let $P : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$ be a functor. We define the **coend** of P to be the space $\coprod_{c \in \mathcal{C}} P(c,c)/\sim$ where the equivalence relation $\sim$ is generated by $P(c,c) \ni P(f^{op}, \mathrm{id})(x) \sim P(\mathrm{id}, f)(x) \in P(d,d)$ where $x \in P(d,c)$. In general the coend can be described as the coequalizer of the diagram

$$\coprod_{f:c\to d} P(d,c) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} P(c,c)$$

Dually, we define the **end** by the subspace of $\prod_{c\in\mathcal{C}} P(c,c)$ of those x for which for all $f : c \to d$ we have $P(\mathrm{id}, f)(x_c) = P(f^{op}, \mathrm{id})(x_d)$ which can similarly be described by an equalizer.

We denote coends by integral signs with index at the top (like $\int^{c\in\mathcal{C}} P(c,c)$), and ends using integral sight with index at the bottom. When the context is clear we will not denote the ambient category or the variables in the functors.

One special situation will be so prevalent that we will use special notation for it: when $F : \mathcal{C}^{op} \to \mathcal{S}, G : \mathcal{C} \to \mathcal{S}$ we write $F \otimes G$ for the coend $\int^c F(c) \times G(c)$. Also, when $F : \mathcal{D} \times \mathcal{C}^{op} \to \mathcal{S}$ and G as before we can take the functor $D \mapsto F(D, -) \otimes G$, which we will also denote by $F \otimes G$. The same situation can occur with G, so in certain cases $F \otimes G$ may be a functor with two arguments. The "tensor" notation we use is unfortunately also used for monoidal categories; while the tensor product does give a monoidal structure, we will not study it. Monoidal structures will not be used in this section, so all tensor symbols denote will the here-defined tensor product.

Examples:

- Let ∇ be the functor $\Delta \to \mathrm{Top}, [n] \mapsto \Delta^n$ where Δ^n is the standard n-simplex. Let X be a simplicial set, and remember that simplicial sets are functors $\Delta \to \mathrm{Set}$. Then we recover the geometric realization of X:

$$X \otimes \nabla = \int^{[n]\in\Delta} X_n \times \Delta^n = |X|$$

- Let F, G be functors $\mathcal{C} \to \mathcal{S}$. The set of natural transformations from F to G is precisely the end $\int_c \mathcal{D}(F(c), G(c))$. A natural transformation is for all $c \in \mathcal{C}$ a function $f_c : F(c) \to G(c)$ such that for $g : c \to d$ we have $G(c) \circ f_c = f_d \circ F(g)$; this is a direct translation of giving an element of $\prod_C \mathcal{D}(F(c), G(c))$ which satisfies the condition given in the definition of an end.

- Since ends are equalizers on products and coends are coequalizers on coproducts in a sense, they are limits and colimits respectively. We therefore get the identities

$$\mathcal{S}(\int^c P(c,c), D) = \int_c \mathcal{S}(P(c,c), D)$$

and

$$\int_c \mathcal{S}(P, D(c)) = \mathcal{S}(P, \int_c D(c))$$

- The integral notation suggest that there is some similarity between ordinary integrals and (co)ends. One similarity is that we have a property which is often named the "Fubini theorem":

$$\int_{(c,e)} F = \int_c \int_e F = \int_e \int_c F$$

We will not utilize this fact, but it gives some light as to why we use this notation. The "Fubini" term comes from analysis where a similar statement holds for integrals; this is one of the reasons for the use of the integral sign.

- If we define $\hom_{\mathcal{S}}(X,Y) = \int_c \hom(X(c), Y(c))$, we get

$$\begin{aligned} &\hom_{\mathcal{S}}(X \otimes_{\mathcal{D}} Y, Z) \\ &= \int_c \mathcal{S}(\int^d X(c,d) \times Y(d), Z(c)) \\ &= \int_{c,d} \mathcal{S}(X(c,d) \times Y(d), Z(c)) \\ &= \int_{c,d} \mathcal{S}(Y(d), \mathcal{S}(X(c,d), Z(c))) \\ &= \int_d \mathcal{S}(Y(d), \int_c \mathcal{S}(X(c,d), Z(c))) \\ &= \hom_{\mathcal{D}}(Y, \hom_{\mathcal{S}}(X,Z)) \end{aligned}$$

Here we used the above properties. This functor is often called the *internal homomorphism* functor. If we combine this statement with the first item we get a proof of Proposition 2.16.

- We have that $* \otimes X \cong \operatorname{colim} X$. A similar statement holds for limits using the internal homomorphism functor described above. This means that all colimits are coends, and we have in turn defined coends by colimits. The concept of a coends does not add anything that we could not have defined before because of this, but it turns out that many definitions and proofs can be stated more compactly using the concept of (co)ends and their properties we describe in this chapter.

The following lemma is often called the co-Yoneda lemma, or sometimes the "Ninja Yoneda lemma":

Lemma 3.9. *Let $X : \mathcal{C} \to \mathrm{Set}, \alpha : \mathcal{D} \to \mathcal{C}$ be functors. Then we get a natural isomorphism*

$$X \circ \alpha \cong \mathcal{C}(\alpha(-), -) \otimes X$$

Proof. Let Y be any set. We denote morphism spaces in sets by $[-,-]$. We have that $[\mathcal{C}(\alpha(d), -) \otimes X, Y] = [\int^c \mathcal{C}(\alpha(d), c) \times X(c), Y]$ by definition. By the fact that coends are colimits dual to ends this is equivalent to $\int_c [\mathcal{C}(\alpha(d), c) \times X(c), Y]$, which by the exponential law in sets is equivalent to $\int_c [\mathcal{C}(\alpha(d), c), [X(c), Y]]$. However, ends over morphism spaces in sets correspond to natural transformations, so this is equivalent to $\mathrm{Set}^{\mathcal{C}}(\mathcal{C}(\alpha(d)), -), [X(-), Y])$. The first of these two functors is exactly the Yoneda functor, so by the Yoneda lemma we get that this is $[X(\alpha(d), U]$. Since all the steps above were isomorphisms which are natural in d, we can again use the Yoneda lemma to get that $\mathcal{C}(\alpha(d), -) \otimes X \cong X(\alpha(d))$ as desired. $\square$

The co-Yoneda term comes from the fact that when one dualizes the above statement to the end case one gets $\int_c \mathbf{Set}(\mathcal{C}(\alpha(d), c), X(c)) \cong X(\alpha(d))$; this uses that the $\otimes$-operation is dual to the morphism-set functor. This is, combined with the natural transformation-set described as an end, precisely the normal Yoneda lemma.

Remark 3.10. In terms of coends the formula for the product of $\mathcal{I}$-spaces reads

$$X \boxtimes Y = \int^{(\mathbf{n},\mathbf{m})} \mathcal{I}(\mathbf{n} \sqcup \mathbf{m}, -) \times X(\mathbf{n}) \times Y(\mathbf{m})$$

This follows from the general formula $\mathrm{Lan}_F G = \int^c \mathcal{D}(F(c), -) \times G(c)$ in an appropriate setting. Using this description the statements about associativity and commutativity take the flavour of pure coend-manipulation.

The coend-formula mentioned above allows us to define a monoidal structure on any category of diagram spaces or sets, as long as the indexing category is monoidal. For a monoidal category $\mathcal{C}, \sqcup$ we define:

$$X \boxtimes Y = \int^{(c,c')} \mathcal{C}(c \sqcup c', -) \times X(c) \times X(c')$$

This general product is known as **Day convolution**. It is a very convenient product as we will find out; one of its more abstract properties that we will not discuss is that it makes the Yoneda functor monoidal:

$$\begin{aligned} &(\mathcal{C}(a, -) \boxtimes \mathcal{C}(b, -))(e) \\ = \int^{(c,c')} &\mathcal{C}(c \sqcup c', e) \times \mathcal{C}(a, c) \times \mathcal{C}(b, c') \\ \cong \int^{c} &\mathcal{C}(c \sqcup b, e) \times \mathcal{C}(a, c) \\ \cong &\mathcal{C}(a \sqcup b, e) \end{aligned}$$

Here we used the Fubini theorem and the co-Yoneda lemma.

Remark 3.11. There is a general result about the Day convolution product: (commutative) monoids in $\mathrm{Set}^{\mathcal{C}}$ are equivalent to (symmetric) lax monoidal functors $\mathcal{C} \to \mathrm{Set}$. A proof of this fact and a more elaborate treatment of the Day convolution can be found in section 9.8 of [Ric20].

The way we have introduced (co)ends above is not the usual definition but rather a consequence of it. The usual definition exhibits a (co)end as a universal construction. We will give the definition for coends as these will be more prevalent in the rest of this text. It is mentioned for completeness, but will not be used in this text.

Definition 3.12. Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, and $f : d \to d'$ a morphism in $\mathcal{C}$. We call w a **cowedge** for F if there is a family of maps $d \to F(c, c)$ for $c \in \mathcal{C}$ which make the following diagram commute for any $f : d \to d'$ a morphism in $\mathcal{C}$:

$$\begin{tikzcd}
F(d', d) \arrow[r, "F(f,\mathrm{id})"] \arrow[d, "F(\mathrm{id},f)"] & F(d, d) \arrow[d] \\
F(d', d') \arrow[r] & w
\end{tikzcd}$$

We define the coend of F to be an initial cowedge, i.e. for any other cowedge w' we have a map $w \to w'$ which commutes with the diagram we have given above for all f.

A very useful property of (co)ends is the "freshman's dream":

Lemma 3.13. *Let $\mathcal{C}$ be a category where the product functor preserves colimits. Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}, G : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{E}$ be functors. Then*

$$\int^{(c,c',d,d') \in \mathcal{C}^2 \times \mathcal{D}^2} F(c, c') \times G(d, d') = (\int^{(c,c')} F(c, c')) \times (\int^{(d,d')} G(d, d'))$$

Proof. We have that

$$\int^{(c,c',d,d') \in \mathcal{C}^2 \times \mathcal{D}^2} F(c, c') \times G(d, d') = \int^{(c,c')} \int^{(d,d')} F(c, c') \times G(d, d')$$

This is due to the Fubini theorem. By the fact that the product preserves colimits and thus coends, and the fact that F is mute in $\mathcal{D}$ we can say

$$\int^{(c,c')} \int^{(d,d')} F(c, c') \times G(d, d') = \int^{(c,c')} (F(c, c') \times \int^{(d,d')} G(d, d'))$$

By the same fact on the other half of the formula we can get the result. $\square$

The main class of examples of categories where the product preserves colimits is formed by cartesian closed categories. Note that since we can express $\operatorname{colim} X = * \otimes X$ it follows that categories where coends are preserved by products have all colimits preserved by products.

3.3 Homotopy colimits

We now introduce homotopy colimits. We can define these using either simplicial sets or topological spaces, but we will mainly use the simplicial variant. As stated in the conventions we assume all non-named categories to be locally small, and from here on we assume them to be small as well. This means that any category we take a homotopy colimit over is small.

The idea behind homotopy colimits is that ordinary colimits are not well-behaved with respect to homotopy theory. For example the pushout does not respect weak equivalence; an example of this is the pushout $* \sqcup_{S^n} D^{n+1}$ of disk and a point along a sphere. This pushout is S^{n+1} which one can visualize for $n = 0$ by joining the ends of an interval to make a circle and for $n = 1$ by pinching the edges of a disk shut to make a sphere. The space D^{n+1} is contractible so weakly equivalent to $*$, so if the pushout would preserve weak equivalence we would expect this to be the same as or at least weakly equivalent to $* \sqcup_{S^n} *$. The latter is of course just a point, so the pushout does not behave well enough for our purposes. This example and more background on homotopy colimits can be found in [Dug].

Definition 3.14. A **bisimplicial** set is a simplicial object in the category of simplicial sets.

The **diagonal** of a bisimplicial set X is the simplicial set $[n] \mapsto (X_n)_n$, and is denoted $d(X)$.

By what was stated in Remark 2.7 we can write bisimplicial sets using double indexes.

Definition 3.15. A **simplicial space** is a simplicial object in the category of topological spaces.

The **geometric realization** of a simplicial space X is given by

$$\int^{[n]\in\Delta} X_n \times \Delta_n$$

This is the same formula as for simplicial sets, except that now X_n has a given topology. When X_n is discrete this coincides with the definition of geometric realization of a simplicial set.

In what follows we will give a simultaneous treatment for diagrams of simplicial sets or topological spaces. For both we will look at a simplicial object in their categories; these form respectively bisimplicial sets or simplicial spaces, and we will for convenience call both these simplicial spaces. The treatment for simplicial sets will use the diagonal, and the variant for spaces will use the realization of a simplicial space. The following lemma justifies why we can take these different approaches:

Lemma 3.16. *Let X be a bisimplicial set. Then realization of the simplicial set $|d(X)|$ is equivalent to the realization of the simplicial space $[n] \mapsto |X_n|$.*

A proof of this statement can be found in section X.8 of [Ric20], or the more original on page 94 of [Qui73]. The proof comes down to the fact that we can show this for representable bisimplicial spaces. By the fact that bisimplicial spaces are presheaves and thus colimits of representables this completes the proof.

In what follows let X be a functor $\mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is either the category of simplicial sets or topological spaces as usual.

Definition 3.17. The **simplicial replacement** of X is the simplicial space $\mathrm{srep}(X)$ defined by

$$\mathrm{srep}(X)_n = \coprod_{a_0 \leftarrow \ldots \leftarrow a_n} X(a_n)$$

Here the $a_0 \leftarrow \ldots \leftarrow a_n$ are chains of maps in $\mathcal{A}$.

If $\mathcal{S} = \mathrm{sSet}$, we define

$$\operatorname*{hocolim}_{\mathcal{A}} X = d(\mathrm{srep}(X))$$

If $\mathcal{S} = \mathrm{Top}$, we instead define

$$\operatorname*{hocolim}_{\mathcal{A}} X = |\mathrm{srep}(X)|$$

For both cases we will also use the shorthand $\mathrm{hocolim}_{\mathcal{A}} X = X_{h\mathcal{A}}$

Note the dependency on the chain $a_n \to \ldots \to a_0$; we will denote elements of the homotopy colimit by tuples $(a_n \to \ldots \to a_0, x)$. A simplicial structure is in both cases defined using the chains similar to how the simplicial structure on the nerve of a category is defined, but the direction of the chains is indeed reversed as the above notation suggests. The only notable distinction we need to make is the case

$$d_n(a_n \xrightarrow{f} a_{n-1} \ldots \to a_0, x) = (a_{n-1} \to \ldots \to a_0, X(f)(x))$$

The bisimplicial structure on $\mathrm{srep}(X)$ for X a simplicial set follows by taking defining the second simplicial structure by $f(a_n \to \ldots \to a_0, x) = (a_n \to \ldots \to a_0, f(x))$. We will not work with this simplicial structure, so we will not define a notation for it. The simplicial structure on the diagonal is given by $f(a_n \to \ldots \to a_0, x) = (f(a_n \to \ldots \to a_0), f(x))$ where the simplicial action on chains is defined as above.

Examples:

- The homotopy colimit over a constant diagram is the nerve: $\mathrm{hocolim}_{\mathcal{A}} * = N\mathcal{A}$. This is clear in the simplicial case, and the topological case it follows from the observation that $\mathrm{srep}(*) = N\mathcal{A}$.

- It is in general much harder to give explicit computations of homotopy colimits of diagrams we are interested in. If we take for example the $\mathcal{I}$ space $\mathbf{n} \mapsto X^n$ for some space X, we get that its homotopy colimit is the space $\coprod_n N(\tilde{\Sigma}_n) \times_{\Sigma_n} X^n$, which is the free algebra induced by the Barratt-Eccles operad. We will look at this some more later, and a full account can be found in [Sch07].

Remark 3.18. There is an alternative formula for homotopy colimits in terms of coends. It is the original definition of the homotopy colimit given by Bousfield and Kan and in the simplicial case it is

$$\operatorname*{hocolim}_{\mathcal{A}} X = \int^{a} N(a \downarrow \mathcal{A}) \times X(a)$$

In the topological case we use the classifying space instead of the nerve. This gives us a way to use the techniques of coends to manipulate homotopy colimits.

This characterisation of the homotopy colimit can be related to the ordinary colimit in the context of model categories; to be precise it gives a clearer way to show that the homotopy colimit is the derived functor of the colimit. A proof of this fact can be found in [Gam10]; another overview of the theory surrounding this fact can be found in [Shu06].

The following is the justification for constructing the homotopy colimit, as described in the beginning of this section:

Proposition 3.19. *Let $X, Y : \mathcal{A} \to \mathcal{S}$ be diagrams and $\eta : X \Rightarrow Y$ a natural transformation which is a levelwise weak equivalence. Then the induced map $X_{h\mathcal{A}} \to Y_{h\mathcal{A}}$ is a weak equivalence.*

We will not give a proof of this statement here. A proof can be found in proposition 4.7 of [Dug], but the statement there requires that the diagrams are objectwise cofibrant. This turns out to not matter in our context: it is proven as theorem A.7 of [DI04] that the cofibrancy condition can be dropped for topological spaces. The standard model structure on simplicial sets has the property that all objects are cofibrant, so for our case we do not need to worry about objects being cofibrant.

We will now exhibit the homotopy colimit as the classifying space or nerve of a category. This category turns out to have some additional structure.

Definition 3.20. A **topological category** is a small category $\mathcal{C}$ with a topology on its set of objects and on the set of all morphisms in $\mathcal{C}$, denoted $\mathrm{Mor}(\mathcal{C})$, such that the composition operation is continuous. Additionally the two maps $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ for the domain and codomain should be continuous. A functor of topological categories is a functor which is continuous on objects and morphism sets. The category of topological categories is denoted $\mathrm{cat}_{\mathcal{S}}$.

A **simplicial category** is analogously a small category with the structure of a simplicial set on objects and morphisms, and the same functors as a above should be morphisms of simplicial sets.

The category of topological categories or simplicial categories is denoted $\mathrm{cat}_{\mathcal{S}}$.

Lemma 3.21. *The nerve of a topological category $\mathcal{C}$ can be given the structure of a simplicial space. The nerve of a simplicial category $\mathcal{D}$ has the structure of a bisimplicial set.*

Proof. The 0-chains in $N\mathcal{C}$ have the topology of the objects of $\mathcal{C}$, the 1-chains have the topology of the morphism sets, and the n-chains have the subspace topology on the n-fold product of the morphism sets. The surface and degeneracy maps are all either compositions or giving the (co)domain, and these we have assumed to be continuous.

The simplicial case is completely analogous. $\square$

This construction is a slight generalisation of the classical nerve and classifying space we have seen before; when we take the topology/simplicial structure to be discrete, we get the usual classifying space and nerve. All properties of the classical construction that we are interested in can also be proven for this construction, so we will tacitly assume these.

Proposition 3.22. *Let $\mathcal{A}$ be a small category and $X : \mathcal{A} \to \mathcal{S}$ a diagram. Then there is a topological category denoted $\mathcal{A}(X)$ such that*

$$\operatorname*{hocolim}_{\mathcal{A}}(X) = |N\mathcal{A}(X)|$$

Note that we see $N\mathcal{A}(X)$ as a simplicial space and that $|-|$ thus means realization of a simplicial space. This proposition also holds in the context of simplicial sets, where we instead get $d(N\mathcal{A}(X))$.

Proof. The objects of $\mathcal{A}(X)$ are the disjoint union of the $X(a), a \in \mathcal{A}$. We will denote an element of this set by (a, x), where $a \in \mathcal{A}$ is the index in the disjoint union and $x \in X(a)$. A morphism $(a, x) \to (a', x')$ consists of a morphism $f : a \to a'$ in $\mathcal{A}$ such that $X(f)(x) = y$.

The topology on objects is given by the coproduct topology, and in the simplicial case it is also given by the levelwise coproduct. The morphism sets are given the subspace topology as a subspace $\mathcal{A}(X)((a, x), (a', x')) \subset \mathcal{A}(a, a') \times X(a)$. Note that this topology is not considered when taking the classifying space

It suffices to show that $N\mathcal{A}(X) = \mathrm{srep}(X)$ in the topological setting. An element of $N\mathcal{A}(X)$ is of the form $(a_0, x_0) \leftarrow ... \xleftarrow{f} (a_n, x_n)$ where $f : a_n \to a_{n-1}$. There is redundant data in this; we know that $x_{n-1} = X(f)(x_n)$, so we can construct an isomorphism sending this to $(a_1 \leftarrow ... \leftarrow a_n, x_n)$. The latter is an element of $\mathrm{srep}(X)$. In the setting where $\mathcal{S} = \mathrm{sSet}$ similar argument applies; an element of $N\mathcal{A}(X)$ of bidegree n, m is then again a chain in $\mathcal{A}$ of length n with an element of $X(a_0)_m$; taking $n = m$ gives the homotopy colimit. $\square$

Remark: the notation $X(\mathcal{A})$ would be more fitting; after all one applies X to all elements of $\mathcal{A}$. We will, however, follow the literature with the notation as above.

If we take the homotopy colimit forming the classical classifying space, so that $X(a) = *$ constantly, we see that $\mathcal{A}(X)$ is a disjoint union of of $\mathcal{A}$ objects, with morphisms $(a, *) \to (b, *)$ for morphisms $a \xrightarrow{f} b$ with $X(f)(*) = *$ which is of course true as all morphisms are mapped to the identity. This means that $\mathcal{A}(X)$ is just $\mathcal{A}$, and we recover $\operatorname{hocolim}_{\mathcal{A}} X = B\mathcal{A}$ as expected.

3.4 Algebra structure on homotopy colimits

We now relate the notion of commutativity of $\mathcal{I}$-spaces, which is weaker than being levelwise commutative, to the notion of an E_∞-space, which in turn is weaker than being commutative.

Theorem 3.23. *Let X be a commutative $\mathcal{I}$-space monoid. Then $X_{h\mathcal{I}}$ is an algebra of the Barratt-Eccles operad.*

This theorem is a consequence of the more general proposition below, which can be found as proposition 6.5 in [Sch09].

Proposition 3.24. *Let O be any operad, and let $\mathcal{E}$ denote the Barratt-Eccles operad. The functor* $\operatorname{hocolim}_{\mathcal{I}}$ *induces a functor $\mathcal{S}^{\mathcal{I}}[O] \to \mathcal{S}[O \times \mathcal{E}]$.*

Proof. Let X be an $\mathcal{I}$-space with an action of O. We can identify $\operatorname{hocolim}_{\mathcal{I}} X$ with $B\,\mathcal{I}(X)$. We need to define a map $\theta : O(k) \times \tilde{\Sigma_k} \times \mathcal{I}(X)^k \to \mathcal{I}(X)$. We can then apply the classifying space functor or nerve functor, which are monoidal, to get the required action. View $O(k)$ in this setting as a topological category, with objects the set $O(k)$ and only identity morphisms. The topology is given by the topology on $O(k)$; it is thus a discrete category, but not necessarily a discrete topological space. Let θ denote its action on X. We define

$$\theta(c, \sigma, (\mathbf{n}_1, x_1), ..., (\mathbf{n}_k, x_k)) = (\mathbf{n}_{\sigma^{-1}(1)} \sqcup ... \sqcup \mathbf{n}_{\sigma^{-1}(k)}, \sigma(n_1, ..., n_k)_* \theta(c, x_1, ..., x_k))$$

To make this a functor and thus a proper action, we also need to define what needs to happen with morphisms. The category $O(k)$ only has identity morphisms which we will ignore. The morphisms of $\tilde{\Sigma_k}$ are of the form $\tau\sigma^{-1} : \sigma \to \tau$. Let $\alpha_i : \mathbf{n}_i \to \mathbf{m}_i$ be morphism in $\mathcal{I}(X)$. We then define, writing α for the sequence,

$$\theta(\tau\sigma^{-1}, \alpha) = \tau\sigma^{-1}(m_{\sigma^{-1}(1)}, ..., m_{\sigma^{-1}(k)})_*(\alpha_{\sigma^{-1}(1)} \sqcup ... \sqcup \alpha_{\sigma^{-1}(k)})$$

We can then write out, writing $y_1 = X(\alpha_i)(x_1)$:

$$\begin{aligned}
&\theta(\tau\sigma^{-1}, \alpha)(\theta(c, \sigma, (\mathbf{n}_1, x_1), ..., (\mathbf{n}_k, x_k))) \\
&= \theta(\tau\sigma^{-1}, \alpha)((\mathbf{n}_{\sigma^{-1}(1)} \sqcup ... \sqcup \mathbf{n}_{\sigma^{-1}(k)}, \sigma(n_1, ..., n_k)_* \theta(c, x_1, ..., x_k)) \\
&= \tau\sigma^{-1}(m_{\sigma^{-1}(1)}, ..., m_{\sigma^{-1}(k)})_*((\mathbf{m}_{\sigma^{-1}(1)} \sqcup ... \sqcup \mathbf{m}_{\sigma^{-1}(k)}, \sigma(m_1, ..., m_k)_* \theta(c, y_1, ..., y_k)) \\
&= (\mathbf{m}_{\tau^{-1}(1)} \sqcup ... \sqcup \mathbf{m}_{\tau^{-1}(k)}, \tau(m_1, ..., m_k)_* \theta(c, y_1, ..., y_k)) \\
&= \theta(c, \tau, (\mathbf{m}_1, y_1), ..., (\mathbf{m}_k, y_k)) \\
&= \theta((\mathrm{id}_{O(k)} \times \tau\sigma^{-1} \times \alpha)(c, \sigma, (\mathbf{n}_1, x_1), ..., (\mathbf{n}_k, x_k))
\end{aligned}$$

where we used that $\sigma^{-1}(\tau\sigma^{-1})^{-1}) = \tau^{-1}$, and that θ commutes with actions on the terms it applies to. Applying the classifying space functor completes the proof. ∎

The theorem follows by taking O to be the commutative operad; $\mathcal{I}$-spaces with a commutative operad action are commutative $\mathcal{I}$-spaces, and the commutative operad action in this case is just the map

$$\theta : X(\mathbf{n}_1) \times ... \times X(\mathbf{n}_k) \to X(\mathbf{n}_1 \sqcup ... \sqcup \mathbf{n}_k)$$

which we get from the fact that X is commutative; note that the fact that X is commutative guarantees that we do not need to worry about whether this is associative. We will denote this map by $(x_1, .., x_k) \mapsto \mu(x_1, ..., x_k)$. The Barratt-Eccles action on the level of categories now becomes

$$\theta(\sigma, (\mathbf{n}_1, x_1), ..., (\mathbf{n}_k, x_k)) = (\mathbf{n}_{\sigma^{-1}(1)} \sqcup ... \sqcup \mathbf{n}_{\sigma^{-1}(k)}, \mu(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(k)}))$$

4 A commutative $\mathcal{I}$-space model of permutative categories

In this section we will study the relationship between permutative categories and commutative $\mathcal{I}$-spaces. More precisely, we will look at a commutative $\mathcal{I}$-space related to a permutative category such that the homotopy colimit of this $\mathcal{I}$-space is the nerve/classifying space of the permutative category. We will then discuss how the E_∞-structures we get from the nerve/classifying space on the one hand and the homotopy colimit of a commutative $\mathcal{I}$-space on the other are related. We first start with some seemingly unrelated general facts which will later all come together when we have defined the commutative $\mathcal{I}$-space we are after.

4.1 $\mathcal{I}$-categories and cofinal functors

We know what $\mathcal{I}$-spaces are for topological spaces and simplicial sets; we similarly define a **$\mathcal{I}$-category** to be a functor $\mathcal{I} \to \mathbf{cat}$ where **cat** denotes the category of small categories. A monoid in this category is again an $\mathcal{I}$-category X with maps $X(\mathbf{n}) \times X(\mathbf{m}) \to X(\mathbf{n} \sqcup \mathbf{m})$ and unit $\mathbf{1} \to X(\mathbf{0})$ where $\mathbf{1}$ is the terminal category. These maps must satisfy the associativity and unitality relations which we have studied before.

Lemma 4.1. *Let X be an $\mathcal{I}$-category. Then $N \circ X$ and $B \circ X$ are $\mathcal{I}$-spaces, and the same holds for (commutative) monoids in the respective categories.*

Proof. The functors N and $|-|$ are lax monoidal, so for instance we get a map

$$N(X(\mathbf{n})) \times N(X(\mathbf{m})) \to N(X(\mathbf{n}) \times X(\mathbf{m})) \xrightarrow{N(\mu)} N(X(\mathbf{n} \sqcup \mathbf{m})).$$

$\square$

Next we will define homotopy cofinal functors; these are functors that induce weak equivalences on homotopy colimits.

Definition 4.2. A functor $F : \mathcal{A} \to \mathcal{B}$ is **homotopy cofinal** if for every $C \in \mathcal{B}$, the space $B(C \downarrow F)$ is contractible.

We denote by $\mathcal{I}_+$ the full subcategory of $\mathcal{I}$ consisting of the elements corresponding to the elements ≥ 1. We denote the inclusion map by i.

Proposition 4.3. *The functor $i : \mathcal{I}_+ \to \mathcal{I}$ is homotopy cofinal.*

Proof. We need to prove that $B(\mathbf{n} \downarrow i)$ is contractible for all $\mathbf{n} \in \mathcal{I}$, or equivalently for $n \geq 1$ that $B(\mathbf{n} \downarrow \mathcal{I}_+)$ is contractible. The category $\mathbf{n} \downarrow \mathcal{I}_+$ has an initial object $\mathrm{id}_\mathbf{n}$ because for any $f : \mathbf{n} \to \mathbf{m}$ we have the unique morphism in $\mathbf{n} \downarrow \mathcal{I}_+$ given by the diagram

$$\begin{tikzcd} & \mathbf{n} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ \mathbf{n} \arrow[rr, "f"] & & \mathbf{m} \end{tikzcd}$$

Categories with an initial object have contractible classifying space so we are done. This does not work for the category $\mathbf{0} \downarrow i$ as $\mathbf{0}$ is not in $\mathcal{I}_+$ and there is no morphism $\mathbf{0} \to \mathbf{0}$. However, it is easily seen that $\mathbf{0} \downarrow i$ is equivalent to $\mathcal{I}_+$ as $\mathbf{0}$ has only unique morphisms to elements in the image of i (which is equivalent to $\mathcal{I}_+$). It thus suffices to show that $\mathcal{I}_+$ is contractible. We do this in the next lemma, which is kept separate as the contractibility of $\mathcal{I}_+$ is used explicitly later on. □

Lemma 4.4. *The category $\mathcal{I}_+$ is contractible*

Proof. We know that $\mathcal{I}$ is contractible as it has the initial object $\mathbf{0}$. We will look at the inclusion map $i : \mathcal{I}_+ \to \mathcal{I}$. The second map is $j : \mathcal{I} \to \mathcal{I}_+, \mathbf{n} \mapsto \mathbf{1} \sqcup \mathbf{n}$ which, when $a : \mathbf{0} \to \mathbf{1}$ is the unique map, is given by $a \sqcup \mathrm{id}_\mathbf{n}$. Both compositions are related to the identity by a natural transformation induced by j. For one half we get $\eta : \mathrm{id}_{\mathcal{I}_+} \Rightarrow i \circ j$ given by concatenation with $\mathrm{id}_\mathbf{1}$. Naturality is given by the square

$$\begin{tikzcd} \mathbf{n} \arrow[r, "i \circ j"] \arrow[d, "f"] & \mathbf{1} \sqcup \mathbf{n} \arrow[d, "\mathrm{id}_\mathbf{1} \sqcup f"] \\ \mathbf{m} \arrow[r, "i \circ j"] & \mathbf{1} \sqcup \mathbf{m} \end{tikzcd}$$

We can similarly give a natural transformation $\eta : \mathrm{id}_\mathcal{I} \Rightarrow j \circ i$. Since natural transformations induce homotopies this means that $B\mathcal{I}$ is homotopy equivalent to $B\mathcal{I}_+$. □

The proof above works for the category $\mathbf{n} \downarrow \mathcal{I}_+$, which is not to be confused with the full subcategory of $\mathcal{I}$ consisting of those $\mathbf{m}$ where $m \geq n$. Those categories do not have an initial object as there are too many morphisms out of each object. The category $\mathbf{n} \downarrow \mathcal{I}_+$ consists of the same objects but paired with morphisms out of $\mathbf{n}$ where we can form this initial object. Note that the choice of the identity in the proof could also have been replaced by any automorphism $\mathbf{n} \to \mathbf{n}$ as all these are isomorphic in $\mathbf{n} \downarrow \mathcal{I}_+$.

This fact can be found as corollary 5.9 of [SS12] in a slightly more general context. The proof there is more involved because it also holds for a different category $\mathcal{J}$ where the proof we have given does not apply.

Proposition 4.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be homotopy cofinal, and $X : \mathcal{A} \to \mathcal{S}$. Then* $\operatorname{hocolim}_{\mathcal{A}} X$ *and* $\operatorname{hocolim}_{\mathcal{B}} X \circ F$ *are weakly equivalent.*

This will turn out to be a consequence of Corollary 4.19, once we have developed some more tools.

4.2 A lemma of Quillen

Now we will work up to a lemma of Quillen which will prove very useful, in that it allows us to compare a homotopy colimit over a functor with one of the images of the functor. We need another definition before we can state the lemma:

Definition 4.6. Suppose we have a commutative diagram of spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

Then the diagram is **homotopy cartesian** or a **homotopy pullback** if for any factorisation $f = B \xrightarrow{i} E \xrightarrow{p} D$ where i is an acyclic cofibration and p a fibration, we have that the map $A \to C \times_D E$ is a weak equivalence.

It turns out that this definition can be loosened a bit; any factorization of f will suffice to show the definition holds, and it is also sufficient to show the property using a factorization of g.

Homotopy pullbacks are quite useful in our context, as the normal fact that pullbacks of isomorphisms are isomorphisms translate into our setting now:

Lemma 4.7. *If*

$$\begin{tikzcd} A \arrow[r, "\overline{g}"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is homotopy cartesian and g is a weak equivalence then $\overline{g}$ is a weak equivalence.

For the proof it should be noted that this definition of a homotopy cartesian square is also applicable to a larger class of model categories; the necessary property which both **Top** and **sSet** enjoy is that they are *right proper*. This means that pullbacks along a fibration preserve weak equivalences (the notion of *left proper* means that pushouts along cofibrations preserve weak equivalences). We need this property for the proof of the lemma:

Proof. Let the square be as in the lemma, and $f = B \xrightarrow{i} E \xrightarrow{p} D$ be the factorization as in the definition. We can draw the following diagram:

$$\begin{tikzcd}
A \arrow[rrr, "\overline{g}"] \arrow[dr, "\sim"] \arrow[dd] & & & B \arrow[dl, "\sim"'] \arrow[dd, "f"] \\
& C \times_D E \arrow[r, "a"] \arrow[dl] & E \arrow[dr, "p"] & \\
C \arrow[rrr, "g"] & & & D
\end{tikzcd}$$

We can then say the arrow a is a weak equivalence, as it is a pullback along the fibration p of the weak equivalence g. It then follows that the $\overline{g}$ is a weak equivalence by the two-out-of-three property. □

It is clear from the definition that it is not sensible to speak of "the" homotopy pullback of a span; it is only unique up to weak equivalence.

The following is a step in the proof of Quillen's theorem B, and appears in [Qui73]. The version we are interested in can be found in [GJ09] IV.5.2, where one can also find its proof. The proof given there relies on Quillen's small object argument, which is not in the scope of this text. We will therefore not discuss a proof of this fact.

Theorem 4.8. *Let $X : \mathcal{A} \to \boldsymbol{sSet}$ be an $\mathcal{A}$-space such that for all $f : a \to b \in \mathcal{A}$ we have that $X(f) : X(a) \to X(b)$ is a weak equivalence, then the following is a homotopy pullback square for all $a \in \mathcal{A}$:*

$$\begin{tikzcd}
X(a) \arrow[r] \arrow[d] & \operatorname{hocolim} X \arrow[d] \\
* \arrow[r, "j_a"] & N\mathcal{A}
\end{tikzcd}$$

Here $*$ is the one-point space, explicitly given by $[n] \mapsto \{*\}$. The map j_a sends the point in $*([n])$ to the n-chain formed by the n-fold composition of id_a. The right vertical map is the map induced by the maps $X(i) \to *$.

The theorem above also holds in the context of topological spaces. This follows because Quillen equivalences preserve homotopy cartesian squares; the details of this are not in the scope of this text.

4.3 Another way of getting E_∞-structures

In what follows, let $(\mathcal{K}, \otimes, e)$ be a permutative small category. We have already seen that $B\mathcal{K}$ naturally has an E_∞-structure. We have also seen that homotopy colimits of commutative $\mathcal{I}$-spaces have E_∞-structures. What we will construct is a commutative $\mathcal{I}$-space associated to $\mathcal{K}$ whose homotopy colimit is related by a zig-zag of weak equivalences to $B\mathcal{K}$, and we will prove that these maps are morphisms of E_∞-spaces.

Definition 4.9. The **rectification** along $\mathcal{I}$ is a functor denoted $\Phi_{\mathcal{I}}(\mathcal{K}) : \mathcal{I} \to$ **cat**. The image of an object $\mathbf{n}$ is defined by $\mathrm{Obj}(\Phi_{\mathcal{I}}(\mathcal{K})(\mathbf{n})) = \mathrm{Obj}(\mathcal{K}^n)$ the n-fold cartesian product, with

$$\Phi_{\mathcal{I}}(\mathcal{K})(\mathbf{n})((k_1, ..., k_n), (l_1, ..., l_n)) = \mathcal{K}(k_1 \otimes ... \otimes k_n, l_1 \otimes ... \otimes l_n)$$

We take $\mathcal{K}^0$ to be the category with one element $*$. The image of a morphism $\alpha : \mathbf{n} \to \mathbf{m}$ is a functor $\alpha_* : \Phi_{\mathcal{I}}(\mathcal{K})(\mathbf{n}) \to \Phi_{\mathcal{I}}(\mathcal{K})(\mathbf{m})$ given on objects by

$$\alpha_*(k_1, ..., k_n) = (k_{\alpha^{-1}(1)}, ..., k_{\alpha^{-1}(m)})$$

where we put e on indices which are not in the image of α. The image of a morphism $f : (k_1, ..., k_n) \to (l_1, ..., l_n)$ is defined as the unique morphism $k_{\alpha^{-1}(1)} \otimes ... \otimes k_{\alpha^{-1}(n)} \to l_{\alpha^{-1}(1)} \otimes ... \otimes l_{\alpha^{-1}(n)}$ which is induced by the isomorphism $k_{\alpha^{-1}(1)} \otimes ... \otimes k_{\alpha^{-1}(n)} \to k_1 \otimes ... \otimes k_n$ and the analogous version for $(l_1, ..., l_n)$; this isomorphism exists because $\mathcal{K}$ is permutative.

More details on this construction can be found in [SS16] and [Sch18].

An immediate observation is that $\Phi_{\mathcal{I}}\mathcal{K}(\mathbf{1}) = \mathcal{K}$. We will denote $N\Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n}) = N_{\mathcal{I}}\mathcal{K}(\mathbf{n})$, and also $B\Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n}) = B_{\mathcal{I}}\mathcal{K}(\mathbf{n})$

Proposition 4.10. *The $\mathcal{I}$-spaces $B_{\mathcal{I}}\mathcal{K}$ and $N_{\mathcal{I}}\mathcal{K}$ have a commutative $\mathcal{I}$-space monoid structure induced by the the permutative structure of $\mathcal{K}$.*

Proof. It is easiest to prove that the $\Phi_{\mathcal{I}}\mathcal{K}$ form a commutative $\mathcal{I}$-category monoid; Lemma 4.1 then gives the rest. For $\mu : \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n}) \times \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{m}) \to \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n} \sqcup \mathbf{m})$ we define on objects $\mu((k_1, ..., k_n), (l_1, ..., l_m)) = (k_1, ..., k_n, l_1, ..., l_m)$ and on morphisms $\mu(f, g) = f \otimes g$. The element in $\Phi_{\mathcal{I}}\mathcal{K}(\mathbf{0}) = \{*\}$ acts as a unit, as we define $\mu(*, x) = \mu(x, *) = x$. For this to be a commutative $\mathcal{I}$-category we need the following diagram from section 3.1 to commute:

$$\begin{tikzcd}
\Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n}) \sqcup \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{m}) \arrow[r, "\mu"] \arrow[d, "\mathrm{swap}"] & \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n} \sqcup \mathbf{m}) \arrow[d, "\Phi_{\mathcal{I}}\mathcal{K}(\chi_{n,m})"] \\
\Phi_{\mathcal{I}}\mathcal{K}(\mathbf{m}) \sqcup \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n}) \arrow[r, "\mu"] & \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{m} \sqcup \mathbf{n})
\end{tikzcd}$$

The map $\chi : \mathbf{n} \sqcup \mathbf{m} \to \mathbf{m} \sqcup \mathbf{n}$ is the block swap. We do have this structure, as according to the definition the swapping action is given by $(\chi_{n,m})_*(k_1, ..., k_n, l_1, ..., l_m) = (l_1, ..., l_m, k_1, ..., k_n)$ which commutes with the swapping of the input on the right. $\square$

Proposition 4.11. *For all $\boldsymbol{n}, \boldsymbol{m} \in \mathcal{I}$ with $n, m > 0$ and $\mathcal{K}$ a permutative category, we have that $B_{\mathcal{I}}\mathcal{K}(\boldsymbol{n})$ and $B_{\mathcal{I}}\mathcal{K}(\boldsymbol{m})$ are weakly equivalent.*

Proof. By the above it suffices to give an equivalence of categories on the underlying categories. For this we also look at the map $j : \mathbf{1} \to \mathbf{n}, j(1) = 1$, inducing a map $\mathcal{K} = \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{1}) \to \Phi_{\mathcal{I}}\mathcal{K}(\mathbf{n})$. This is part of an equivalence of categories; we take the map $m : (k_1, ..., k_n) \mapsto k_1 \otimes ... \otimes k_n$. For the composite $m \circ j$ of an element k, note that $j(k)$ is just the sequence $(k, e, ..., e)$ with only k at index 1, and the image under m of this is just k, so $m \circ j$ is the identity on $\mathcal{K}$. For $j \circ m$, note that $(j \circ m)(k_1, ..., k_n) = (k_1 \otimes ... \otimes k_n, e, ..., e)$, which is naturally isomorphic to $(k_1, ..., k_n)$. $\square$

We can make this statement somewhat more specific:

Corollary 4.12. *For all morphisms $f : \boldsymbol{n} \to \boldsymbol{m}$ in $\mathcal{I}_+$, we have that the induced map $f_* : B_{\mathcal{I}}\mathcal{K}(\boldsymbol{n}) \to B_{\mathcal{I}}\mathcal{K}(\boldsymbol{m})$ is a weak equivalence*

Proof. Define $j_k : \mathbf{1} \to \mathbf{n}, j(1) = k$ for $1 \leq k \leq n$. Note that the choice of using j_1 in the proof of the proposition above is more of a convenience than a necessity; the proof will work with any j_k. Note that $f \circ j_k = j_{f(k)}$, so we also get that $f_* \circ (j_k)_* = (j_{f(k)})_*$. Both $(j_k)_*$ and $(j_{f(k)})_*$ are weak equivalences by the above proposition, so by the two-out-of-three property of weak equivalences we get that f_* is a weak equivalence. $\square$

All the work we have done in the last two sections culminates in the following statement. It can be found in a slightly different context in the proof of Proposition 4.18 of [SS16].

Theorem 4.13. *We have a weak equivalence $N\mathcal{K} \to \operatorname{hocolim}_{\mathcal{I}} N_{\mathcal{I}}\mathcal{K}$ or $B\mathcal{K} \to \operatorname{hocolim}_{\mathcal{I}} B_{\mathcal{I}}\mathcal{K}$ induced by the inclusion $\{\boldsymbol{1}\} \to \mathcal{I}$.*

Proof. By the fact that $\mathcal{I}_+ \subset \mathcal{I}$ is homotopy cofinal, we only have to prove that there is a weak equivalence $B\mathcal{K} \to \operatorname{hocolim}_{\mathcal{I}_+} N_{\mathcal{I}}\mathcal{K}$. This follows from the fact that $\mathcal{I}_+$ is contractible as proven in Lemma 4.4, so $* \to N\mathcal{I}_+$ is a weak equivalence. As $f : \mathbf{i} \to \mathbf{j} \in \mathcal{I}_+$ induce a weak equivalence $B_{\mathcal{I}}\mathcal{K}(\mathbf{i}) \to B_{\mathcal{I}}\mathcal{K}(\mathbf{j})$, the conditions of Theorem 4.8 above holds and by Lemma 4.7 it is a weak equivalence. $\square$

4.4 The theorem of Hollender-Vogt

Definition 4.14. Let $\mathcal{S}$ be the category of spaces or simplicial sets as before. Let $\mathcal{C}$ be a small category, $X : \mathcal{C}^{op} \to \mathcal{S}, Y : \mathcal{C} \to \mathcal{S}$. We the define the **total bar construction** of X and Y along $\mathcal{C}$ to be the bisimplicial set or simplicial space

$$[n] \mapsto \coprod_{c_0 \to ... \to c_n} X(c_0) \times Y(c_n)$$

The **bar construction** for simplicial sets is the diagonal of this, and the version for topological spaces takes the geometric realization. Both versions will be denoted $B(X, \mathcal{C}, Y)$.

Similar to the case for the tensor operation we can take $X : \mathcal{D} \times \mathcal{C}^{op} \to \mathcal{S}, Y : \mathcal{C} \times \mathcal{E} \to \mathcal{S}$ and get $B(X, \mathcal{C}, Y) : \mathcal{D} \times \mathcal{E} \to \mathcal{S}$. This also signifies why we choose to explicitly denote the category along which the bar construction is taken.

One detail we skipped over are the face and degeneracy maps. They work similarly to those in the homotopy colimit: we shorten or extend the chain of morphisms in the middle. Specifically, the face maps insert an identity, while the degeneracy maps compose adjacent morphisms. The only exceptions to this rule are the edge cases. We work in $B(Y, \mathcal{C}, X)$, with $f_n \circ ... \circ f_1 : c_n \to ... \to c_0$, $y \in Y(c_n), x \in X(c_0)$:

$$d^0(f_n \circ ... \circ f_1, y, x) = (f_n \circ ... \circ f_2, y, X(f_1)(x))$$
$$d^n(f_n \circ ... \circ f_1, y, x) = (f_{n-1} \circ ... \circ f_1, Y(f_n)(y), x)$$

At this point it becomes clear why we demand Y to be contravariant; to get an element of $Y(c_{n-1})$ we need to move y in the reverse direction of the arrow f_n, which can be done if Y is contravariant.

The bar construction will form a more general tool than both the tensor product defined above and the homotopy colimits we study. At a first glance it is already clear that $B(*, \mathcal{C}, *) = N\mathcal{C}$, and it is also easy to see that $B(*, \mathcal{C}, X) = \operatorname{hocolim}_{\mathcal{C}} X$. This means that if we study properties of the bar construction we can sometimes translate these to properties of homotopy colimits.

A more accurate relation between the bar construction and the tensor operation is that the bar construction is the derived tensor operation. This matches the relation to the colimit; recall that $* \otimes_{\mathcal{C}} X = \operatorname{colim}_{\mathcal{C}} X$, and that $B(*, \mathcal{C}, X) = \operatorname{hocolim}_{\mathcal{C}} X$. Details on this relationship can be seen in for example Section 21 of [Shu06].

The following are some of the properties of the bar construction. The main resource for this section is [HV92] as this leads to the result we want, but the reader should beware that this article does suppress some notation such as $\mathcal{C}(-, F(-))$ being denoted by $\mathcal{C}$. Another useful resource is [Shu06], which does however not go exactly in the direction we need.

We will silently assume that $\mathcal{C}$ is topologically enriched or a topological category. This means that $\mathcal{C}(a, b)$ can be assumed to be a topological space. We will not delve too deeply into this structure for now.

Lemma 4.15. *We have the following natural homotopy equivalences, where $X : \mathcal{C}^{op} \to \mathcal{S}, Y, Z : \mathcal{C} \to \mathcal{S}$. These also work when we take $X : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}$ and similar for Y, Z to get natural isomorphims instead of isomorphism*

1. $B(X, \mathcal{C}, Y) \otimes Z \cong B(X, \mathcal{C}, Y \otimes Z)$
2. $B(B(X, \mathcal{C}, Y'), \mathcal{D}, Z) \cong B(X, \mathcal{C}, B(Y', \mathcal{D}, Z))$ *where* $Y' : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}$ *and* $Z : \mathcal{D} \to \mathcal{S}$
3. *There is a natural homotopy equivalence $B(\mathcal{C}, \mathcal{C}, Y) \simeq Y$; this is the analogue of the co-Yoneda lemma.*

4. *$B(X,\mathcal{C},Y) \cong B(Y^{op},\mathcal{C}^{op},X^{op})$, where $Y^{op}:\mathcal{C}^{op}\to\mathcal{S}$ with $Y^{op}(c) = c, Y^{op}(f^{op}:d\to c) = Y(f:c\to d)$.*

Proof. 1. $B(X,\mathcal{C},Y)\otimes Z = \coprod_a \coprod_{c_0\to\ldots\to c_n} X(c_n)\times Y(a,c_0)\times Z(a)/\sim$. The equivalence relation does not act on the term $X(c_n)$, so we can leave this out of the relation. It is also unaffected by a, so using the distributive law for products and coproducts this reduces to $\coprod_{c_0\to\ldots\to c_n} X(c_n)\times (\coprod_a Y(a,c_0)\times Z(a)/\sim) = B(X,\mathcal{C},Y\otimes Z)$

2. We write $\coprod_c$ for $\coprod_{c_0\to\ldots\to c_n}$ for brevity:

$$\begin{aligned} &B(B(X,\mathcal{C},Y'),\mathcal{E},Z') \\ =&\coprod_d(\coprod_c X(c_n)\times Y'(c_0,-)(d_n)\times Z(d_0) \\ =&\coprod_{d,c} X(c_n)\times Y'(c_0,d_n)\times Z(d_0) \\ =&\coprod_c X(c_n)\times(\coprod_d Y'(-,d_n)\times Z(d_0))(c_0) \\ =&B(X,\mathcal{C},B(Y',\mathcal{D},Z)) \end{aligned}$$

3. The map we will use is the map $\phi:(g,f_1\circ\ldots\circ f_n,x)\mapsto X(g\circ f_1\circ\ldots f_n)(x)$. Let $r:X\to B(\mathcal{C},\mathcal{C},X)$ be the map $x\mapsto(\mathrm{id},\mathrm{id}\circ\ldots\circ\mathrm{id},x)$. We construct an explicit simplicial homotopy H between the identity and $r\circ\phi$. We define this by

$$H_m(n,(g,f_1\circ\ldots\circ f_m,x)) = (g,\mathrm{id}\circ\ldots f_1\circ\ldots\circ f_{m-n},X(f_{m-n+1}\circ\ldots\circ f_m)(x))$$

with at $n=0$ doing nothing, and at $n=m+1$ we apply g as well to arrive at the image of $r\circ\phi$. One can check that this defines a simplicial homotopy.

For naturality of ϕ, let h be a suitable morphism in $\mathcal{C}$. Then we can write

$$(\phi\circ h_*)(g,f_1\circ\ldots\circ f_n,x) = \phi(h\circ g,f_1\circ\ldots\circ f_n,x) = X(h)\phi(g,f_1,\ldots,f_n,x)$$

which proves that ϕ is natural.

4.

$$\begin{aligned} &B(Y^{op},\mathcal{C}^{op},X^{op}) \\ =&\coprod_{c_0\leftarrow\ldots\leftarrow c_n} X^{op}(c_0)\times Y^{op}(c_n) \\ =&\coprod_{c_0\leftarrow\ldots\leftarrow c_n} X(c_0)\times Y(c_n) \\ =&B(X,\mathcal{C},Y) \end{aligned}$$

□

Note that the fourth statement of this lemma allows us to exchange items in the other three statements; for instance that $B(X, \mathcal{C}, \mathcal{C}) \cong X$. Be aware that the functors do change direction, so the example we gave technically follows from $B(\mathcal{C}^{op}, \mathcal{C}^{op}, X^{op}) \cong X^{op}$.

Recall that for $d \in \mathcal{D}$ we have the category $d \downarrow \mathcal{D}$ of arrows out of d. If $F : \mathcal{C} \to \mathcal{D}$ is a functor we define $\mathcal{D} \downarrow F$ to be the category with objects (f, c) with $f : d \to F(c)$ and the natural maps.

Lemma 4.16. *We have a homotopy equivalence* $B(*, \mathcal{C}, \mathcal{D}(-, F(-))) \simeq N(\mathcal{D} \downarrow F)$

Proof. Morphisms in $D \downarrow F$ from (C_0, f_0) to (C_1, f_1) are given by a morphism $g : C_0 \to C_1$ such that $F(g) \circ f_0 = f_1$. Therefore, given $(C_0, f_0) \in D \downarrow F$ and $g : C_0 \to C_1$ we can uniquely make from this a $(C_1, f_1) \in D \downarrow F$ with g inducing a morphism $(C_0, f_0) \to (C_1, f_1)$: we take $f_1 = F(g) \circ f_0$, which satisfies what we want by definition. We pick an element

$$(f_1 \circ ... \circ f_n : C_0 \to C_n, \alpha : D \to F(C_0)) \in B(*, \mathcal{C}, \mathcal{D}(D, F(-))$$

We associate to this the chain

$$(C_0, \alpha) \to (C_1, F(f_1) \circ \alpha) \to (C_2, F(f_2 \circ f_1) \circ \alpha) \to ...$$

by the process described above; this exactly gives an element of the nerve of $D \downarrow F$. Naturality is quite straightforward, and we do not write it out. □

Corollary 4.17. *We get a weak equivalence* $\operatorname{hocolim}_{\mathcal{C}} X \simeq B(\mathcal{C} \downarrow \mathrm{id}) \otimes X$

Proof.

$$\begin{aligned} \operatorname{hocolim} X = B(*, \mathcal{C}, X) \simeq \\ B(*, \mathcal{C}, \mathcal{C} \otimes X) \cong \\ B(*, \mathcal{C}, \mathcal{C}(-, \mathrm{id}(-))) \otimes X \simeq \\ B(\mathcal{C} \downarrow \mathrm{id}) \otimes X \end{aligned}$$

Here we used the co-Yoneda lemma Lemma 3.9 and the interplay between the bar construction and the tensor product from Lemma 4.15, combined with the previous lemma. □

The formula as in the above corollary is how Bousfield and Kan originally defined the homotopy colimit. We can offer yet another formula for the homotopy colimit:

$$\operatorname*{hocolim}_{\mathcal{C}} X = B(*, \mathcal{C}, X) = B(* \otimes \mathcal{C}, \mathcal{C}, X) = * \otimes B(\mathcal{C}, \mathcal{C}, X) = \operatorname{colim} B(\mathcal{C}, \mathcal{C}, X)$$

This version, which is taken from [HV92], exhibits the homotopy colimit as a colimit of a certain resolution of X, and can in practice give computable formulas for homotopy colimits. We call $B(\mathcal{C}, \mathcal{C}, X)$ the **bar resolution** of X. The following lemma gives another description of this resolution, which is used in [SS16].

Lemma 4.18. *Let π denote the natural transformation $\mathcal{C} \downarrow \mathrm{id} \to \mathrm{const}_{\mathcal{C}}, c \to d \mapsto c$. We view $B(*, \mathcal{C} \downarrow \mathrm{id}, X \circ \pi)$ as a functor $\mathcal{C} \to \mathcal{S}$ using the functor to small categories in the middle position; the natural transformation π makes that we can functorially use the bar construction. We have a weak equivalence $B(*, \mathcal{C} \downarrow \mathrm{id}, X \circ \pi) \to B(\mathcal{C}, \mathcal{C}, X)$.*

Proof. The proof is similar to the above; a chain in $(\mathcal{C} \downarrow \mathrm{id})(C)$ consists of a chain in $\mathcal{C}$ of objects with morphisms to C, which all commute. This is equivalent to giving a chain in $\mathcal{C}$ with a morphism from the end of the chain to C. The functor X is only applied to the first object of the chain and not to the morphism of the first object, so the total data we have is: a chain $c_0 \to .. \to c_n$ in $\mathcal{C}$, an element of $\mathcal{C}(c_n, C)$, and an element of $X(c_0)$. This is just the same as giving an element of $B(\mathcal{C}, \mathcal{C}, X)$ $\square$

Corollary 4.19. *We get the following properties of homotopy colimits:*

- $\mathrm{hocolim}_{\mathcal{C}} B(X, \mathcal{D}, Y) \simeq B(\mathrm{hocolim}_{\mathcal{C}} X, \mathcal{D}, Y)$, *where* $X : \mathcal{C} \times \mathcal{D}^{op} \to \mathcal{S}$
- *The cofinality theorem, i.e. if $B(D \downarrow F)$ is contractible for all $D \in \mathcal{D}$, then we have a weak equivalence $X_{h\mathcal{D}} \simeq (X \circ F)_{h\mathcal{C}}$.*

Proof. The first statement follows from substituting the bar-version of the homotopy colimit. For the cofinality theorem, note that by assumption

$$* \cong B(\mathcal{D} \downarrow F) = B(*, \mathcal{C}, \mathcal{D}(-, F(-))$$

We can then compute

$$\begin{aligned} X_{h\mathcal{D}} &= B(*, \mathcal{D}, X) \\ &\simeq B(B(\mathcal{D} \downarrow F), \mathcal{D}, X) \\ &\simeq B(B(*, \mathcal{C}, \mathcal{D}(-, F(-)), \mathcal{D}, X) \\ &\cong B(*, \mathcal{C}, B(\mathcal{D}(-, F(-)), \mathcal{D}, X)) \\ &= B(*, \mathcal{C}, B(\mathcal{D}, \mathcal{D}, F) \circ F) \\ &\simeq B(*, \mathcal{C}, B(*, \mathcal{D}, X \circ F)) \\ &= (X \circ F)_{h\mathcal{C}} \end{aligned}$$

$\square$

Definition 4.20. Let $X : \mathcal{C} \to \mathcal{S}, F : \mathcal{C} \to \mathcal{D}$. We define the **homotopy left Kan extension** of X along F by $B(\mathcal{D}(F(-), -), \mathcal{C}, X)$, and denote it by $F_{h^*}X$

From this point on we will call these homotopy Kan extensions; homotopy right Kan extensions exist, but we will not need them in this text.

In the case of ordinary (left) Kan extensions we have that $\mathrm{Lan}_F \circ \mathrm{Lan}_G X = \mathrm{Lan}_{F \circ G} X$, which can be seen by the fact that Lan_F is left adjoint to precomposition with F. Additionally, if F is a functor $\mathcal{C} \to *$ then $\mathrm{Lan}_F X \cong \mathrm{colim}_{\mathcal{C}} X$; combining these facts gives the formula $\mathrm{colim}(\mathrm{Lan}_G X) = \mathrm{colim}\, X$. The homotopy version of this statement is the following:

Proposition 4.21. *There is a natural weak equivalence*

$$\operatorname*{hocolim}_{\mathcal{D}} F_{h^*}X \simeq \operatorname*{hocolim}_{\mathcal{C}} X$$

Proof. We can write out

$$\begin{aligned} &\operatorname*{hocolim}_{\mathcal{D}} F_{h^*}X \\ &= B(*, \mathcal{D}, B(\mathcal{D}(F(-), -), \mathcal{C}, X) \\ &\cong B(B(*, \mathcal{D}, \mathcal{D}(F(-), -)), \mathcal{C}, X) \end{aligned}$$

We have that $B(*, \mathcal{D}, \mathcal{D}(F(-), -))$ as a functor $\mathcal{C} \to \mathcal{S}$ is the composition $B(*, \mathcal{D}, \mathcal{D}) \circ F$, but we have that $B(*, \mathcal{D}, \mathcal{D}) \simeq *$, so we also get a natural weak equivalence

$$B(*, \mathcal{D}, \mathcal{D}(F(-), -)) \simeq *$$

This proves the proposition. □

One should be aware of the difference between the proof above and the situation as in Lemma 4.16; there we were looking at $\mathcal{D}(-, F(-))$ and here we looked at $\mathcal{D}(F(-), -)$. This difference is vital to this argument, as in this case F was not applied to an element of the chain inside the bar construction, but to the argument for the bar construction as a whole.

4.5 The Grothendieck construction and Thomason's theorem

For the next step in our construction, we will refine our Proposition 3.22 some more. We will be looking specifically at homotopy colimits of the form $(NF)_{h\mathcal{C}}$, where F is a $\mathcal{C}$-diagram of categories. We want to get a construction on such a diagram such that the nerve of this construction is the homotopy colimit we are interested in. In other words we could say that we want something which behaves like the homotopy colimit within the category of small categories. If we denote this by $\mathcal{C}\int F$ (where F is a $\mathcal{C}$-diagram), we want a weak equivalence:

$$N(\mathcal{C}\textstyle\int F) \simeq (NF)_{h\mathcal{C}}$$

Let us look at the right side of this equation, and give an element of this, say of degree m. This is a chain $c_0 \xrightarrow{j_1} c_1 \to \ldots \xrightarrow{j_n} c_n$ in $\mathcal{C}$, and a chain $k_0 \xrightarrow{f_1} k_1 \to \ldots \xrightarrow{f_n} k_n$ in $F(c_0)$. Note that in the homotopy colimit these chains have equal length, as the homotopy colimit of simplicial sets is defined as the diagonal of a bisimplicial set. We want to exhibit this data as the diagonal row of a big 'diagram', which we could draw like the following (writing $(-)^*$ for for the action of F).

$$\begin{tikzcd}
k_0 \arrow[r] \arrow[d] \arrow[dr, dashed] & j_1^* k_0 \arrow[r] \arrow[d] & \dots \arrow[r] & (j_n \circ \dots \circ j_1)^* k_0 \arrow[d] \\
k_1 \arrow[r] \arrow[d] & j_1^* k_1 \arrow[r] \arrow[d] \arrow[dr, dashed] & \dots & \dots \\
\dots \arrow[d] & \dots & \dots \arrow[dr, dashed] & \dots \\
k_m \arrow[rr] & & \dots & (j_n \circ \dots \circ j_1)^* k_m
\end{tikzcd}$$

Not that this is strictly not a diagram of arrows in a category, as there are no arrows from k_0 to $j_1^* k_0$; one is in $F(c_0)$ and the other in $F(c_1)$, and these are generally different categories. We remedy this by gluing all the categories together.

Definition 4.22. Let $\mathcal{C}$ be a small category, and $F : \mathcal{C} \to \text{cat}$. We define the **Grothendieck construction** of this data to be the category with objects (C, X) where $C \in \mathcal{C}, X \in F(C)$. Morphisms $(C, X) \to (D, Y)$ are given by pairs $(f : C \to D, g : F(f)(X) \to Y)$. Composition is given by $(f, g) \circ (f', g') = (f \circ f', g \circ F(f)(g'))$. We denote this category by $\mathcal{C} \int F$.

A definition of the Grothendieck construction and related concepts can be found in chapter V of [Ric20]. The definition can also be found in [HV92] and [Tho79] for the purpose that we will use it for.

Remark 4.23. The integral notation in the Grothendieck construction is very reminiscent of a (co)end; indeed we have

$$\mathcal{C} \int F = \int^{c \in \mathcal{C}} (c \downarrow \mathcal{C}) \times F(c)$$

The Grothendieck construction solves the problem we had before where some of the arrows in the diagram did not represent any arrows in a category. Now we have an arrow $(c_0, k_0) \to (c_1, j_1^* k_0)$ given by (j_1, id). Also, in the big diagram above all the square commute when one fills in all the appropriate maps. If we take for example to top left square, we have $(\text{id}, j_1^*(f_1)) \circ (j_1, \text{id}) = (j_1, \text{id}) \circ (\text{id}, f_1) = (j_1, f_1)$ by using the definition of composition in this category. Therefore we get a well-defined chain on the diagonal, which is a chain in $N(\mathcal{C} \int F)$ as desired. We denote the map which performs this procedure by η. Writing out fully, we define

$$\eta(c_0 \xrightarrow{j_1} c_1 \to \dots \xrightarrow{j_n} c_n, k_0 \xrightarrow{f_1} k_1 \to \dots \xrightarrow{f_n} k_n) =$$
$$(c_0, k_0) \xrightarrow{(j_1, f_1)} (c_1, j_1^*(k_1)) \to \dots \xrightarrow{(j_n, (j_{n-1} \circ \dots \circ j_1)^*(f_n))} (c_n, (j_n \circ \dots \circ j_1)^*(k_n))$$

The use of this construction is in the fact that η is a weak equivalence. This result was first achieved in [Tho79], where the Grothendieck construction is exhibited as a formulation of the homotopy colimit in the category of small categories. The proof by Thomason gives a functor $\tilde{F}$ and maps

$$\begin{tikzcd} & \operatorname{hocolim} N\tilde{F} \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\ \operatorname{hocolim} NF \arrow[rr, "\eta"] & & N(\mathcal{C}\textstyle\int F) \end{tikzcd}$$

Thomason proves that λ_1, λ_2 are weak equivalences and that there is a homotopy $\eta \circ \lambda_1 \Rightarrow \lambda_2$, by which η is a weak equivalence. We will, however, give a proof formulated in [HV92] by Hollender and Vogt which relies heavily on the machinery of the bar construction. We will later show that this proof does not differ too much from the proof by Thomason by exhibiting that the middle spaces in both proofs use are equivalent.

Theorem 4.24. *Let $F : \mathcal{C} \to \mathrm{cat}$ be a functor. Let $P : \mathcal{C}\int F \to \mathcal{C}, (K, C) \mapsto K$ and $* : \mathcal{C}\int F \to \mathcal{S}$ be the constant one-point space. There are weak equivalences*

$$\operatorname*{hocolim}_{\mathcal{C}} N \circ F \leftarrow \operatorname*{hocolim}_{\mathcal{C}} P_{h^*} * \to N(\mathcal{C}\textstyle\int F)$$

Remark: It might seem plausible that we can get an equivalence between the left and right terms using basic manipulations of coends, as both the homotopy colimits and Grothendieck constructions can be written in this way. If we write the nerve as $N(-) = \mathbf{cat}(\Delta^{op}, -)$ we can write this theorem as

$$\int^{c} \mathbf{cat}(\Delta^{op}, c \downarrow \mathcal{C} \times F(c)) \cong \mathbf{cat}(\Delta^{op}, \int^{c} c \downarrow \mathcal{C} \times F(c))$$

where we have already used that the homomorphism-functor preserves limits in the second argument. It does in general not preserve colimits in the second argument, so there is no simple coend-manipulating proof of this theorem.

Proof. For the right map, we compute:

$$\begin{aligned} &\operatorname*{hocolim}_{\mathcal{C}} P_{h^*} * \\ &= B(*, \mathcal{C}, B(\mathcal{C}(P(-), -), \mathcal{C}\textstyle\int F, *)) \\ &\cong B(B(*, \mathcal{C}, \mathcal{C}(P(-), -)), \mathcal{C}\textstyle\int F, *) \\ &= B(B(*, \mathcal{C}, \mathcal{C}) \circ P, \mathcal{C}\textstyle\int F, *) \\ &\simeq B(* \circ P, \mathcal{C}\textstyle\int F, *) \\ &= N(\mathcal{C}\textstyle\int F) \end{aligned}$$

For the left map, pick $K \in \mathcal{C}$. We make the diagram:

$$\begin{tikzcd} F(K) \arrow[r, "J"] \arrow[d, "*"] & \mathcal{C}\textstyle\int F \arrow[d, "P"] \\ * \arrow[r, "\mathrm{id}_K"] & \mathcal{C} \end{tikzcd}$$

Here J is the inclusion map $C \mapsto (K, C)$. We first finish up the proof using another calculation before finishing up the final details.

$$\textbf{claim:} \mathcal{C}(P(-), K) \simeq B(*, F(K), \mathcal{C} \int F(-, J(-)))$$

This allows us to complete the proof. We compute:

$$\begin{aligned}
&(P_{h^*}*)(K) \\
&= B(\mathcal{C}(P(-), K), \mathcal{C} \int F, *) \\
&\simeq B(B(*, F(K), \mathcal{C} \int F(-, J(-))), \mathcal{C} \int F, *) \\
&\cong B(*, F(K), B(\mathcal{C} \int F(-, J(-)), \mathcal{C} \int F, *)) \\
&\simeq B(*, F(K), * \circ J) \\
&= N(F(K))
\end{aligned}$$

Proof of claim: Let us look at an element of the right hand side (in degree n) applied to $(K', c) \in \mathcal{C} \int F$: it consists of a chain $F(c_0) \to .. \to F(c_n)$ in $F(K)$, and an element $f \in \mathcal{C} \int F((K', c), (K, c_0))$, which corresponds to a morphisms $f : K' \to K$ and a morphism $F(f)(c) \to c_0$. The whole space can therefore be written as a combination of $\mathcal{C}(K', K)$ and spaces of the form $B(*, F(K), F(K)(D, -)$; to be exact we take the pullback of $\mathcal{C}(K', K)$ and the coproduct over all $D \in \mathrm{Obj}(F(K))$ of $B(*, F(K), F(K)(D, -))$. We take the pullback along the set $\mathrm{Obj}(F(K))$, where for the left hand side we take the map $f \mapsto F(f)(c)$, and for the right hand side we take $B(*, F(K), F(K)(D, -)) \mapsto D$. We thus get a pullback diagram

$$\begin{tikzcd}
B(*, F(K), \mathcal{C} \int F(-, J(-))) \arrow[r] \arrow[d] & B(*, F(K), F(K)(D, -)) \arrow[d] \\
\mathcal{C}(K', K) \arrow[r] & \mathrm{Obj}(F(K))
\end{tikzcd}$$

The map on the right hand side is a homotopy equivalence (as each component is), and since pullbacks of homotopy equivalences are homotopy equivalences the space we started with is weakly equivalent to $\mathcal{C}(K', K) = \mathcal{C}(P(K', c), K)$ as desired. $\square$

In the proof by Thomason in [Tho79], the middle space is given by $\mathrm{hocolim}_{\mathcal{C}} N\tilde{F}$, where $\tilde{F}(c) = P \downarrow c$ with P as above. This makes no difference:

Proposition 4.25. *The following two spaces are isomorphic:*

$$\underset{\mathcal{C}}{\mathrm{hocolim}}\, P_{h^*} * \cong \underset{\mathcal{C}}{\mathrm{hocolim}}\, N\tilde{F}$$

Proof. We write out what they are:

$$\begin{aligned} &(\operatorname*{hocolim}_{\mathcal{C}} P_{h^*}*)_n \\ = &\coprod_{c_0\to\ldots\to c_n} B(\mathcal{C}(P(-),-),\mathcal{C}\textstyle\int F,*)_n \\ = &\coprod_{c_0\to\ldots\to c_n}\ \coprod_{(d_0,x_0)\to\ldots\to(d_n,x_n)} \mathcal{C}(d_n,c_0) \end{aligned}$$

On the other hand we get:

$$\begin{aligned} &\operatorname*{hocolim}_{\mathcal{C}} N\tilde{F} \\ = &\coprod_{c_0\to\ldots\to c_n} N(P\downarrow c_0)_n \\ = &\coprod_{c_0\to\ldots\to c_n} \{(d_0,x_0)\to\ldots\to(d_n,x_n), d_n\to c_0\} \end{aligned}$$

Here we used that a chain in $P\downarrow c_0$ only requires the last morphism to c_0 as data, as the others can be inferred by the necessary commuting triangles. The two spaces we have given are obviously the same, which proves our claim. □

This proposition gives us a very useful tool: we can use the proof given by Hollender and Vogt while retaining the explicit map given by Thomason.

4.6 Maps of E_∞-algebras

In this section we will study the results we have seen in the context of E_∞-algebras, as the spaces we studied in the previous section can in some cases be E_∞-spaces. The question we want to answer is whether the weak equivalences we have shown actually preserve this structure.

Lemma 4.26. *Let X be a commutative $\mathcal{K}$-category monoid, with $\mathcal{K}$ a permutative category. Then $B(\mathcal{K}\int X)$ has an E_∞-space induced by the permutative structure of $\mathcal{K}$ and the commutative monoid structure of X.*

Proof. The Barratt-Eccles operad for categories is formed by the categories $\tilde{\Sigma}_n$ with objects for the permutations and a morphism between every two permutations. Denote $(\mathbf{n},x)$ for an element of $\mathcal{C}\int X$. Then we can define

$$\theta_k(\sigma,(\mathbf{n}_1,x_1),...,(\mathbf{n}_k,x_k)) = (\mathbf{n}_{\sigma^{-1}(1)}\sqcup\ldots\sqcup\mathbf{n}_{\sigma^{-1}(k)}, \mu(x_{\sigma^{-1}(1)},...,x_{\sigma^{-1}(k)}))$$

Here we used the map $\mu : X(\mathbf{n})\times X(\mathbf{m})\to X(\mathbf{n}\sqcup\mathbf{m})$ which we get from X being a commutative monoid. Since this operation is associative we are allowed to apply μ to multiple entries without denoting how to parenthesise. The equivariance of this expression is guaranteed by the commutativity of X; if we permute the factors on the left side by τ, then the result is the same when we act by τ on the right hand side.

A similar formula can be applied to morphisms, giving us the structure we require. □

An immediate consequence is that $N(I\int\Phi_{\mathcal{I}}\mathcal{K})$ and $B(I\int\Phi_{\mathcal{I}}\mathcal{K})$ are E_∞-spaces in our setting.

Theorem 4.27. *Let η be the map from Thomason's theorem, where $\mathcal{K}$ is now permutative and F a commutative $\mathcal{K}$-category. Then η preserves the E_∞-structure*

Proof. This is relatively straightforward, but a rather expansive calculation. Let μ denote the multiplication of X under the adjunction as discussed in Section 3.1. Since μ is strictly associative we can put any number of arguments in μ. We use the shorthand $\sqcup^\sigma j = j^{\sigma^{-1}(1)}\sqcup ... \sqcup j^{\sigma^{-1}(m)}$ and $\mu^\sigma(x) = \mu(j^{\sigma^{-1}(1)}, ..., j^{\sigma^{-1}(m)})$ and $\sigma = \sigma_0 \to ... \to \sigma_n$. We compute

$$\begin{aligned}
&\theta_m(\sigma, (\eta(c_0^i \xrightarrow{j_1^i} ... \xrightarrow{j_n^i} c_n, x_1^i \xrightarrow{f_1^i} ... \xrightarrow{f_1^i} x_n^i)_{i=1..m}))\\
&= \theta_m(\sigma, ((c_0^i, k_0^i) \xrightarrow{(j_1^i, f_1^i)} (c_1^i, (j_1^i)^*(x_1^i)) \to ...\\
&\xrightarrow{(j_n^i, (j_{n-1}^i ... j_1^i)^*(f_n^i))} (c_n^i, (j_n^i ... j_1^i)^*(x_n^i))_{i=1..m})\\
&= (\sqcup^\sigma c_0, \mu^\sigma(k_0) \xrightarrow{\sqcup^\sigma j_1, \mu^\sigma f_1} ...\\
&\xrightarrow{(\sqcup^{\sigma_{n-1}} j_n, \mu^{\sigma_{n-1}}(j_{n-1}...j_1)^*(f_n))} (\sqcup^{\sigma_n} c_n, \mu^{\sigma_n}(j_n^i ... j_1^i)^*(x_n^i)))
\end{aligned}$$

On the other hand we get:

$$\begin{aligned}
&\eta(\theta_m(\sigma, (c_0^i \xrightarrow{j_1^i} ... \xrightarrow{j_n^i} c_n, x_1^i \xrightarrow{f_1^i} ... \xrightarrow{f_1^i} x_n^i)_{i=1..m}))\\
&= \eta((\sqcup^\sigma c_0^i \xrightarrow{\sqcup^\sigma j_1^i} ... \xrightarrow{\sqcup^\sigma j_n^i} \sqcup^\sigma c_n, \mu^\sigma x_1^i \xrightarrow{\mu^\sigma f_1^i} ... \xrightarrow{\mu^\sigma f_1^i} \mu^\sigma x_n^i)))\\
&= (\sqcup^\sigma c_0, \mu^\sigma x_0) \xrightarrow{\sqcup^\sigma j_1, \mu^\sigma f_1} ...\\
&\xrightarrow{(\sqcup^{\sigma_n} j_n, ((\sqcup^{\sigma_{n-1}} j_n)\circ ... \circ(\sqcup^{\sigma_{n-1}} j_1))^*(\mu^{\sigma_n} f_n))} (\sqcup^{\sigma_n} c_n,\\
&((\sqcup^{\sigma_{n-1}} j_n) \circ ... \circ (\sqcup^{\sigma_{n-1}} j_1))^*(\mu^{\sigma_n} x_n)
\end{aligned}$$

This would be equal if we could get the following square (where we forget the variables above) to commute for all $\mathbf{n}_i, \mathbf{m}_i, j_i$:

$$\begin{array}{ccc}
X(\mathbf{n}_1) \times X(\mathbf{n}_2) & \xrightarrow{j_1^* \times j_2^*} & X(\mathbf{m}_1) \times X(\mathbf{m}_2) \\
\downarrow \mu & & \downarrow \mu \\
X(\mathbf{n}_1 \sqcup \mathbf{n}_2) & \xrightarrow{(j_1 \sqcup j_2)^*} & X(\mathbf{m}_1 \sqcup \mathbf{m}_2)
\end{array}$$

This square commutes because μ is natural. Furthermore we need $(j_1 \sqcup j_2) \circ (j_1' \sqcup j_2') = j_1 \circ j_2 \sqcup j_1' \circ j_2'$ but this is guaranteed by the fact that $\sqcup$ is a functor. $\square$

As an aside we check that the bar construction we defined earlier behaves well, in the sense that it makes $\mathcal{K}$-space monoids into E_∞-algebras. It turns out to not be necessary for our purposes, but it is nevertheless useful for showing that the proof of Thomason's theorem by Hollender and Vogt in [HV92] still takes place in an E_∞-context.

Proposition 4.28. *Let $\mathcal{K}$ be a permutative category (with product map $\sqcup$), X a commutative $\mathcal{K}$-space monoid and Y a commutative $\mathcal{K}^{op}$-space monoid, we have that $B(Y, \mathcal{K}, X)$ is an algebra over the Barratt-Eccles operad.*

Proof. We check this at level n; suppose $\sigma = \sigma^0 \to ... \to \sigma^n \in N(\tilde{\Sigma}_m)_n$, and for each $i = 1...m$ we have $b_i = (c_0^i \to ... \to c_n^i, y^i, x^i) \in B(Y, \mathcal{C}, X)_n$. Let us define:

$$\begin{aligned} &\theta_m(\sigma, b_1, ..., b_m) \\ &\quad = (\theta_{\mathcal{K}}(\sigma^0, (c_0^i)_{i=1}^n) \to ... \to \theta(\sigma^m, (c_{n_i}^i)_{i=1}^n), \mu_Y(y_1, ..., y_m), \mu_X(x_1, ..., x_m)) \end{aligned}$$

Again the commutativity of X and Y are necessary to make this an equivariant map. The maps between the $\theta_{\mathcal{C}}(\sigma^j, (c_j^i)_{i=1}^m)$ for j and $j+1$ is given by the action of $\theta_{\mathcal{K}}$ on the unique morphism from σ^j to σ^{j+1} and the morphisms between the c_i^j and c_i^{j+1}. □

We can compose the map from Thomason's theorem with the following to get the desired results. This is analogous to proposition 4.18 from [SS16].

Proposition 4.29. *There is a weak equivalence between $N\mathcal{K}$ and $N(\mathcal{I}\int\Phi_{\mathcal{I}}\mathcal{K})$ which is a map of E_∞-algebras.*

Proof. Let us first mention the fact that any $j \in \mathcal{I}(\mathbf{n}, \mathbf{m})$ can be factored as an order-preserving map $\mathbf{n} \to \mathbf{m}$ after a permutation; let us call this permutation $\tilde{j}$. Note that this is in this case gives a permutation on n elements; doing this the other way around would not be well-defined. This construction however is uniquely defined; if we claim to have a different permutation, two elements must be reversed in order, resulting in a different result after the order-preserving map which would be a contradiction. Let $P : \mathcal{I}\int\Phi_{\mathcal{I}}\mathcal{K} \to \mathcal{K}$ be the functor $(\mathbf{n}, (k_1, ..., k_n)) \mapsto k_1 \otimes ... \otimes k_n, (j, f) \mapsto f \circ \tilde{j}^*$. To prove that P is a functor we take $\mathbf{n} \xrightarrow{h} \mathbf{m} \xrightarrow{j} \mathbf{p}$ and

$$g : (k_1, ..., k_m) \to (l_1, ..., l_m), f : (l_{j^{-1}(1)}, ..., l_{j^{-1}(p)}) \to (r_{j^{-1}(1)}, ..., r_{j^{-1}(p)})$$

and look at the composition $(j, f) \circ (h, g)$. We have defined this to be $(j \circ h, f \circ \Phi_{\mathcal{I}}\mathcal{K}(j)(g))$. We can write the second component of this, together with the diagram defining $\Phi_{\mathcal{I}}\mathcal{K}(j)(g)$:

$$\begin{tikzcd} k_1 \otimes ... \otimes k_n \arrow[r, "g"] \arrow[d, "j^*"] & l_1 \otimes ... \otimes l_n \arrow[d, "j^*"] & \\ k_{j^{-1}(1)} \otimes ... \otimes k_{j^{-1}(p)} \arrow[r, "\Phi_{\mathcal{I}}\mathcal{K}(j)(g)"] & l_{j^{-1}(1)} \otimes ... \otimes l_{j^{-1}(p)} \arrow[r, "f"] & r_{j^{-1}(1)} \otimes ... \otimes r_{j^{-1}(p)} \end{tikzcd}$$

Notice that strictly speaking these are maps in the image of $\Phi_{\mathcal{I}}\mathcal{K}$, so should have written the objects as tuples. We have written the underlying maps in $\mathcal{K}$ in this case, and we will not make a notational distinction. Also note that in the bottom row we can write $(k_{j^{-1}(1)} \otimes ... \otimes k_{j^{-1}(p)})$ as $(k_{\tilde{j}^{-1}(1)} \otimes ... \otimes k_{\tilde{j}^{-1}(p)})$ and replace the map j^* by $\tilde{j}^*$ because by definition we put the monoidal units at indices which are not in the image of j and this corresponds to how we have defined $\tilde{j}$. We can then write out the definition

$$P((j,f) \circ (g,h)) = P(j \circ h, f \circ \phi_{\mathcal{I}}\mathcal{K}(j)(g)) = f \circ \phi_{\mathcal{I}}\mathcal{K}(j)(g) \circ (\widetilde{j \circ h})^*$$

We have that $\widetilde{j \circ h} = \tilde{j} \circ \tilde{h}$ by a simple write-out, and using the diagram with the remark above we get:

$$f \circ \phi_{\mathcal{I}}\mathcal{K}(j)(g) \circ (j \tilde{\circ} h)^* = f \circ \phi_{\mathcal{I}}\mathcal{K}(j)(g) \circ \tilde{j}^* \circ \tilde{h}^* = f \circ \tilde{f}^* \circ g \circ \tilde{h}^*$$

This is clearly the same as $P(j,f) \circ P(g,h)$ which makes P a functor.

We can now form the following chain of morphisms:

$$N_{\mathcal{I}}\mathcal{K}(\mathbf{1}) \xrightarrow{i} N_{\mathcal{I}}\mathcal{K}_{h\,\mathcal{I}} \xrightarrow{\eta} N(\mathcal{I}\textstyle\int\Phi_I\mathcal{K}) \xrightarrow{NP} N\mathcal{K}$$

Here η denotes Thomason's map from section 4.5, and i is the inclusion $c \mapsto (\mathbf{1} \to ... \to \mathbf{1}, c)$. We have that η is a weak equivalence, and by Theorem 4.13 i is a weak equivalence (as $N_{\mathcal{I}}\mathcal{K}(\mathbf{1}) = N\mathcal{K}$). If we look at the composition of all these maps we can write out:

$$\begin{aligned} N_{\mathcal{I}}\mathcal{K}(\mathbf{1}) \ni (k_0 \xrightarrow{f_1} ... \xrightarrow{f_n} k_n) &\overset{i}{\mapsto} \\ (1 \xrightarrow{\text{id}} ... \to 1, k_0 \to ... \to k_n) &\overset{\eta}{\mapsto} \\ (\mathbf{1}, k_0) \xrightarrow{(\text{id}, f_1)} ... \xrightarrow{(\text{id}, f_n)} (\mathbf{1}, k_n) &\xmapsto{NP} \\ (k_0 \xrightarrow{f_1} ... \xrightarrow{f_n} k_n)& \end{aligned}$$

This means that this composition is the identity which is is a weak equivalence, and two out of the three maps are weak equivalences. Therefore the third, being NP, must be a weak equivalence.

It is noteworthy that it is not necessary to start with $N_{\mathcal{I}}\mathcal{K}(\mathbf{n})$ where $n = 1$; if we had chosen any other n we would not have had the identity but rather the map from Proposition 4.11 which we have proven is a weak equivalence.

Let us prove that P preserves the algebra structure:

$$\begin{aligned} &P(\theta_m(\sigma, (\mathbf{n}_i, (k_1^i, ..., k_{n_i}^i))_{i=1...m})) \\ = P(\mathbf{n}_{\sigma^{-1}(1)} \sqcup ... \sqcup \mathbf{n}_{\sigma^{-1}(m)}, (k_1^{\sigma^{-1}(1)}, ..., &k_{n_{\sigma^{-1}(1)}}^{\sigma^{-1}(1)}, k_1^{\sigma^{-1}(2)}, ..., k_{n_{\sigma^{-1}(m)}}^{\sigma^{-1}(m)})) \\ = (k_1^{\sigma^{-1}(1)} \otimes ... \otimes &k_{n_{\sigma^{-1}(1)}}^{\sigma^{-1}(1)}, k_1^{\sigma^{-1}(2)} \otimes ... \otimes k_{n_{\sigma^{-1}(m)}}^{\sigma^{-1}(m)}) \\ &= \theta_m(\sigma, P((\mathbf{n}_i, (k_1^i, ..., k_{n_i}^i))_{i=1...m}) \end{aligned}$$

Here the last step can be taken because we can parenthesise all the blocks. □

What we have shown thus far is the following:

Theorem 4.30. *We have a chain of weak equivalence relating $N\mathcal{K}$ and $(N_{\mathcal{I}}\mathcal{K})_{h\,\mathcal{I}}$ which are morphisms of Barratt-Eccles algebras.*

5 Applications to commutative $\mathcal{K}$-space monoids

In this section we will no longer look at just a permutative category $\mathcal{K}$, but broaden the scope to look at $\mathcal{K}$-spaces and $\mathcal{K}$-categories, i.e. $\mathcal{K}$-diagrams of small categories. The category of $\mathcal{K}$-spaces has a monoidal structure constructed in the same way as in the category of $\mathcal{I}$-spaces; after all we only used its permutative structure and nothing specific about this category. The key difference is the lack of understanding what the monoidal product does; in $\mathcal{I}$-spaces we could in examples like the matrix groups get some grasp of what this comes down to, but now there is no a priori intuition. There are other interesting diagrams of interest in the literature; two examples are [SS16], which deals with braided injections, and [SS12], which deals with the category $\mathcal{I}$ and another category named $\mathcal{J}$.

In the rest of this section $\mathcal{K}$ will be a permutative category with monoidal structure $\otimes, e$.

We begin by recalling the result from Theorem 3.23 and making it slightly more general:

Proposition 5.1. *Let X be a commutative $\mathcal{K}$-space monoid. Then $X_{h\mathcal{K}}$ is an algebra over the Barratt-Eccles operad.*

Proof. Write $\mu : X(k) \times X(l) \to X(k \otimes l)$ for the multiplication map under the adjunction as seen in section 3.1. Then we write

$$\theta(\sigma, (k_1, x_1), ..., (k_n, x_n)) = (k_{\sigma^{-1}(1)} \otimes ... \otimes k_{\sigma^{-1}(n)}, \mu(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)})$$

□

Let $\mathcal{K}$ be a small permutative category, and X a commutative $\mathcal{K}$-space. Recall from Proposition 3.22 the notation $\mathcal{K}(X)$; for now we do not care whether we use the simplicial or topological version, and we will avoid using specific properties of these constructions. We will define certain objects in terms of topology to avoid saying "space structure", but one can check that the same can constructions can be carried out using simplicial sets. Recall that we showed that $\mathcal{K}(X)$ is a topological category, and that the category of topological categories is denoted $\mathrm{cat}_{\mathcal{S}}$ with morphism being functors which are continuous as maps of spaces. This is not to be confused with the notion of a 'continuous functor', i.e. a functor preserving all small limits.

Remark 5.2. Note that this construction is not comparable with the Grothendieck construction. The Grothendieck construction looks at a $\mathcal{K}$-category and unifies this into a category. In our case we get a procedure that takes a $\mathcal{K}$-space and makes a topological category.

Proposition 5.3. *The mapping $X \mapsto \mathcal{K}(X)$ is a functor $\mathcal{S}^{\mathcal{K}} \to$ $\boldsymbol{cat}_{\mathcal{S}}$*

Proof. We give the objects of $\mathcal{K}(X)$ a topology by seeing it as the disjoint union of the $X(k)$ as mentioned before. Let $F : X \to Y$ be a morphism of $\mathcal{K}$-spaces, i.e. a natural transformation. Then we define $\mathcal{K}(F)$ to be the functor which takes (k, x) to $(k, F_k(x))$. We will prove that functors induce continuous maps in the topological setting; the simplicial case can be done analogously. Let $\mathcal{K}(F)$ be the image of a functor as above. For continuity we will look at the pre-image of an open set $U \subset \mathcal{K}(Y)$, which we can assume to be contained in $Y(k)$ for some k since all open sets are unions of such open sets. Then the pre-image of U is contained in $X(k)$ since $\mathcal{K}(F)$ does not change which component we are in. But then the pre-image of U are the elements (k, x) with x in the pre-image under F_k of U, which is open as F_k is continuous.

The simplicial case is analogous, and we will not prove it here. $\square$

Proposition 5.4. $\mathcal{K}(X)$ *is permutative category when* X *is a commutative* $\mathcal{K}$-*space monoid, and the monoidal map is continuous as a map of spaces.*

Proof. If $\otimes$ denotes the product in $\mathcal{K}$ and μ the adjoint of the product in X as discussed in section 3.1, we define $(k, x) \oplus (l, y) = (k \otimes l, \mu(x, y)$. This makes sense, as $\mu : X(k) \times X(l) \to X(k \otimes l)$ so this is an element of $\mathcal{K}(X)$. The unit is formed by $(e, *)$ where e is the unit in $\mathcal{K}$ and $*$ is the image of the unit map $\eta : 1 \to X(e)$. Associativity follows from associativity of both operations. The weak commutativity is given by the map $\tau : k \otimes l \to l \otimes k$ from the weak commutativity in $\mathcal{K}$; the fact that $X(\tau)(\mu(x, y)) = \mu(y, x)$ follows from the definition of a commutative $\mathcal{K}$-space where commutativity is exactly expressed by this property.

For the monoidal map we calculate the explicite preimage $\oplus^{-1}(k, x) = \bigcup_{n+m=k} \mu_{n,m}^{-1}(x)$ which is open when applied to an open set because all the $\mu_{n,m}$ are continuous and have inverse images in different components. $\square$

A good reality check is that to see whether this permutative structure is related to the other structure we can get out of $\mathcal{K}(X)$, i.e. the E_∞-structure on the homotopy colimit. This turns out to be the case.

Lemma 5.5. *Let* $N\mathcal{K}(X)$ *have the* E_∞-*structure inherited from the permutative structure on* $\mathcal{K}(X)$. *Then this structure is isomorphic to the structure from the homotopy colimit on* $X_{h\mathcal{K}}$

Proof. Note that we have switched back to the simplicial setting as this makes it easier to write out the algebra structure. The topological version follows as usual. Let $(k_0, x_0) \xrightarrow{f_1} ... \xrightarrow{f_n} (k_n, x_n) \in N\mathcal{K}(X)$; this corresponds to

$$(k_0 \xrightarrow{f_1} ... \xrightarrow{f_n} k_n, x_0) \in X_{h\mathcal{K}}$$

as $x_1 = X(f_1)(x_0)$ etc. For the structure on $N\mathcal{K}(X)$ we compute (again using $\otimes^i$ and μ^i for operations like $\otimes^i x^i = x^1 \otimes ... \otimes x^n$ where the bounds are understood):

$$\begin{aligned}
&\theta_m(\sigma_0 \to ... \to \sigma_n, (k_0^i, x_0^i) \xrightarrow{f_1^i} ... \xrightarrow{f_n^i} (k_n^i, x_n^i)) \\
= &(\otimes^i k_0^{\sigma_0^{-1}(i)}, \mu^i(x_0^{\sigma_0^{-1}(i)}) \xrightarrow{s_1} ... \to s_n(\otimes^i k_n^{\sigma_n(i)}, \mu^i(x_n^{\sigma_n^{-1}(i)})) \\
\mapsto &(\otimes^i k_0^{\sigma_0^{-1}(i)} \xrightarrow{s_1} ... \to s_n \otimes^i k_n^{\sigma_n(i)}, \mu^i(x_0^{\sigma_0^{-1}(i)}) \\
= &\theta_m(\sigma_0 \to ... \to \sigma_n, (k_0^i \xrightarrow{f_1^i} ... \xrightarrow{f_n^i} k_n^i), x_0^i))
\end{aligned}$$

which is the structure on $X_{h\mathcal{K}}$. The maps s_i are the

$$\tau_{\sigma_i \sigma_{i-1}^{-1}} \circ (f_i^{\sigma_{i-1}^{-1}(1)} \times ... \times f_i^{\sigma_{i-1}^{-1}(m)})$$

where the τ arises from the permutative structure on $\mathcal{K}$ $\square$

Lemma 5.6. *The space $B_{\mathcal{I}}(\mathcal{K}(X))$ is a commutative $\mathcal{I}$-space. We denote it* $\mathrm{hocolim}^{\mathcal{I}} X$*. Note that here we take $\mathcal{K}(X)$ as a topological category, and that the classifying space we take here is the classifying space of topological categories.*

Proof. It suffices to show that $\Phi_{\mathcal{I}}(\mathcal{K}(X))(\mathbf{n})$ is a topological category for all $\mathbf{n}$, and that the multiplication is continuous as a functor. The fact that $\mathcal{K}$ is permutative then gives the commutative $\mathcal{I}$-space structure. We give

$$\mathrm{Obj}(\Phi_{\mathcal{I}}(\mathcal{K}(X))(\mathbf{n})) = \mathcal{K}(X)^n$$

the product topology. For the topology on the morphisms we define it as the pullback in the following diagram:

$$\begin{tikzcd}
\mathrm{Mor}(\Phi_{\mathcal{I}}\mathcal{K}(X))(\mathbf{n}) \arrow[r] \arrow[d] & \mathrm{Mor}\mathcal{K}(X) \arrow[d, "{(s,t)}"] \\
\mathrm{Obj}(\mathcal{K}(X))^n \times \mathrm{Obj}(\mathcal{K}(X))^n \arrow[r, "\otimes^n \times \otimes^n"] & \mathrm{Obj}(\mathcal{K}(X)) \times \mathrm{Obj}(\mathcal{K}(X))
\end{tikzcd}$$

Here s, t are the source and target maps respectively. The top horizontal map is the one by which we have defined morphisms in $\Phi_{\mathcal{I}}\mathcal{C}$ for any permutative $\mathcal{C}$. The vertical map on the left then forms the source and target maps respectively. This matches the original definition for morphisms in $\Phi_{\mathcal{I}}\mathcal{C}(\mathbf{n})$ we gave for general permutative categories $\mathcal{C}$; the morphisms there were exactly the morphisms in $\mathcal{C}$ where the source and target were n-fold monoidal products. This definition gives continuous source and target maps by definition. For composition we can draw the following pullback diagram

$$\begin{tikzcd}
\mathrm{Mor}(\Phi_{\mathcal{I}}\mathcal{K}(X))(\mathbf{n}) \times_{\mathrm{Obj}(\mathcal{K}(X))^n} \mathrm{Mor}(\Phi_{\mathcal{I}}\mathcal{K}(X))(\mathbf{n}) \arrow[r] \arrow[d] & \mathrm{Mor}(\Phi_{\mathcal{I}}\mathcal{K}(X))(\mathbf{n}) \arrow[d] \\
\mathrm{Mor}(\mathcal{K}(X)) \times_{\mathrm{Obj}(\mathcal{K}(X))} \mathrm{Mor}(\mathcal{K}(X)) \arrow[r] & \mathrm{Mor}(\mathcal{K}(X))
\end{tikzcd}$$

The pullbacks occurring in the left column signify that the target of one morphism is the source of the next to make the morphisms composable. The vertical maps are continuous by the previous diagram, and the bottom horizontal map is continuous because $\mathcal{K}(X)$ is a topological category. It then follows that the to horizontal map is continuous as well. $\square$

We now come to the main theorem: every commutative $\mathcal{K}$-space can be modelled by a commutative $\mathcal{I}$-space. This means that homotopy colimits over arbitrary permutative categories can be exchanged for homotopy colimits over $\mathcal{I}$

Theorem 5.7. *We have that* $(\mathrm{hocolim}^{\mathcal{I}} X)_{h\mathcal{I}} \simeq X_{h\mathcal{K}}$ *as* E_∞*-algebras*

The proof of this fact is in one line described by the weak equivalences

$$X_{h\mathcal{K}} \cong N\mathcal{K}(X) \simeq N_{\mathcal{I}}(\mathcal{K}(X))_{h\mathcal{I}} = (\mathrm{hocolim}^{\mathcal{I}} X)_{h\mathcal{I}}$$

The first is taken from Lemma 5.5, the second is Theorem 4.30 and the third is by definition. We have proven that all these weak equivalences preserve the E_∞-structure. We have, however, proven Theorem 4.30 in the case of a permutative category while here we have a permutative topological category. We thus need to check that the relevant statements from Proposition 4.10 onward also hold for topological categories. Note that statements such as Theorem 4.13 which only concern $\mathcal{I}$-spaces and not the underlying $\mathcal{I}$-categories are of no concern in this.

- The multiplication map from Proposition 4.10 is continuous because of the definition of the product topology. Note that in this case we are looking at the commutative $\mathcal{I}$-category monoid, and not the commutative $\mathcal{I}$-space monoid we get after taking the classifying space; the continuity of the maps in the latter is guaranteed by the classifying space construction.

- In Proposition 4.11 we have the map j which is continuous by the product topology definition again; the map m is continuous by Proposition 5.4.

- For Definition 4.14 we take a closer look at the definition as in [HV92], which does assume the category over which the bar construction is taken to be a topological category. There the standing definition is that diagrams over a topological category are *continuous*, which in this case means that if X is a $\mathcal{C}$-diagram that then $\mathcal{C}(c, d) \to \mathcal{S}(X(c), X(d))$ is continuous for all c, d. Here $\mathcal{S}(-,-)$ denotes the morphism space in either topological spaces or simplicial sets, where for the topological case one should use a convenient category of topological spaces. By the latter we mean that one should restrict to a category of spaces where the morphism sets again form a space within the category; an example of this is described in section VIII.5 of [Ric20].

  The bar construction $B(X, \mathcal{C}, Y)$ is then defined as the realization of the simplicial space

  $$[n] \mapsto \{(x, f_1, ..., f_n, y) \mid x \in s(f_1), y \in t(f_n)\}$$

  with s, t denoting the source and target maps. The topology is the subspace topology of the product topology. The face and degeneracy maps are the same as we originally defined them, and here we see it is necessary that X and Y have the continuity property; we have $d^0(x, f_1, ...) = (X(f_1)(x), ...)$ and because X is continuous on maps and the evaluation is continuous (see for example VIII.5.15 of [Ric20]) this is continuous. This is a generalisation of the previous definition of the bar construction. All subsequent properties of bar constructions and other definitions can also be found in [HV92] using topological categories, so they hold true.

- The Grothendieck construction of Definition 4.22 needs to be defined for when we have a diagram of topological categories. Let $F : \mathcal{C} \to \mathrm{cat}_{\mathcal{S}}$ be such a diagram. We give $\mathrm{Obj}(\mathcal{C}\int F)$ the coproduct topology. The morphisms are topologized as a subset

$$\mathrm{mor}(\mathcal{C}\textstyle\int F) \subset \mathrm{mor}(\mathcal{C}) \times \displaystyle\coprod_c F(c) \times \coprod_c \mathrm{mor}(F(c))$$

  We now have that we can write $s(f, x, g) = (s(f), x)$ and $t(f, x, g) = (t(f), t(g))$ which both give continuous maps. Composition is given by

$$((f, x, g), (f', x', g')) \mapsto (f \circ f', x, g \circ F(f)(g'))$$

  This is continuous because $F(f)$ is continuous as F is a functor to topological categories, and evaluation is continuous. Note that the original definition does not mention the source object; we need it here to make a sensible source map.

- The statement of Theorem 4.24 is also true in our context. The map P is continuous as $\mathcal{C}$ has the discrete topology and $P^{-1}(K)$ is exactly $F(K)$ which is open and closed. It is continuous on morphisms because on morphisms it is a projection. The codomain of P' is the disjoint union of all $F(K)$ so this maps is also continuous on both objects and morphisms. In the proof we also use a map J which on objects is an inclusion map from the definition of the coproduct and therefore continuous. On morphisms we can write it as $J(g) = (\mathrm{id}_K, s(g), g)$ which is continuous. The rest of the proof uses manipulations of these functors, which by the above is allowed in this context.

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